An Exposition of Generalized Kac-Moody algebras

Elizabeth Jurisich

Abstract. We present a detailed exposition of the theory of generalized Kac-Moody algebras associated to symmetrizable matrices. A proof of the character formula for a standard module is given, generalizing the argument of Garland and Lepowsky for the Kac-Moody case. A short proof of the theorem that any generalized Kac-Moody algebra can be decomposed into direct (vector space) sums of free subalgebras and a Kac-Moody subalgebra is given.

Introduction

This paper provides a detailed exposition of the theory of generalized Kac-Moody algebras, whose theory was originally developed by R. Borcherds. The theory of generalized Kac-Moody Lie algebras has an important application in Borcherds' proof [B3] of the “Monstrous Moonshine” conjectures of Conway and Norton for the moonshine module of [FLM], where the Monster Lie algebra (a generalized Kac-Moody algebra) plays an important role (see also [J] and [JLW]).

It is shown in [J], that generalized Kac-Moody algebras with no mutually orthogonal simple imaginary roots have a decomposition in terms of a Kac-Moody subalgebra and large free subalgebras over certain modules of the Kac-Moody subalgebra. Using this decomposition some of the theory of this type of generalized Kac-Moody algebra becomes simplified (esp. in the case of the Monster Lie algebra, see [J] and [JLW]). As an application of the character formula for standard modules a short proof is provided of a more general theorem of E. Jurisich and R. Wilson [JW] decomposing any generalized Kac-Moody algebra in a similar way.

Results such as character formulas and a denominator identity known for Kac-Moody algebras are stated in [B1] for generalized Kac-Moody algebras. We found it necessary to do some additional work in order to understand fully the precise definitions and also the reasoning which are implicit in Borcherds’ work on this subject. V. Kac in [K] gives an outline (without detail) of how to rigorously develop the theory of generalized Kac-Moody algebras by indicating that one should follow the arguments presented there for Kac-Moody algebras (this is the approach

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of [HMY]). Here we develop the theory of generalized Kac-Moody algebras, generalizing arguments appearing in the exposition [L] (see also [K] and [M]), and by extending the homology results of [GL] (not covered in [K]) to these new Lie algebras. The treatment here closely resembles that in [L] and [GL] and is sometimes essentially identical, but here additional arguments are necessary to handle the presence of simple imaginary roots. Proofs of the character formula for standard modules and the denominator formula for a generalized Kac-Moody algebra associated to a symmetrizable matrix are thus presented. That the results of [GL] hold for generalized Kac-Moody algebras is mentioned and used in [B3]. The work done here generalizing [GL] has also already been carried out in the work of S. Naito [N] (where the matrices are assumed finite).

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1. Definition of a Generalized Kac-Moody Algebra

1.1 Construction of the algebra associated to a matrix.

In [B1] Borcherds defines the generalized Kac-Moody algebra associated to a matrix, which for convenience he takes to be symmetric. Two further definitions (different from the first and from one another) are given in [B2] and [B3]. Of these definitions the one given below most resembles the one appearing in [B1], where there is a definition in terms of generators consisting of $e_i$’s, $f_i$’s, for $i$ in some index set $I$, and a vector space $H$, and relations determined by square matrices which agree with the ones given here. Borcherds does not divide out by the radical (that is, the largest graded ideal having trivial intersection with $h$) in his definition of generalized Kac-Moody algebra.

The Lie algebra denoted $g'(A)$ in [K], which is defined from an arbitrary matrix $A$, is equal to the generalized Kac-Moody algebra $g(A)$, defined below, when the matrix $A$ satisfies conditions C1–C3 given below.

Let $E$ be a field of characteristic 0. Let $F \subset E$ be a subfield which is equal to some subfield of $\mathbb{R}$, for example $E = \mathbb{C}$ or $\mathbb{R}$, $F = \mathbb{R}$ or $\mathbb{Q}$. Vector spaces will be over $E$ unless otherwise indicated.

Let $I$ be a finite or countably infinite set, which we identify with $\{1, 2, \ldots, k\}$ or $\mathbb{Z}_+$ (the positive integers) for notational purposes. Let $A = (a_{ij})_{i,j \in I}$ be a matrix with entries in $E$.

Let $g_F$ be the free Lie algebra on the generators $h_i, e_i, f_i$, where $i \in I$. Let $s$ be the ideal generated by the elements:

\[
\begin{align*}
[h_i, h_j] \\
[h_i, e_k] - a_{ik}e_k \\
[h_i, f_k] + a_{ik}f_k \\
[e_i, f_j] - \delta_{ij}h_i
\end{align*}
\]

for all $i, j, k \in I$.

**Definition.** Let $g_0 = g_F/s$. 


The free associative algebra $F$ over the $h_i,e_i,f_i$ has an associative algebra grading determined by giving $h_i$ degree $(0,0,\ldots)$, $e_i$ degree $(0,0,0,1,0,\ldots)$, and $f_i$ degree $(0,0,0,0,-1,0,\ldots)$, where $\pm 1$ appears in the $i^{th}$ position. Restricting this grading to the natural copy of $g_F$ in $F$ gives $g_F$ a $\mathbb{Z}$-Lie algebra grading. For a sequence of integers $(n_1,n_2,\ldots)$ let $g_F(n_1,n_2,\ldots)$ denote the corresponding subspace of $g_F$. Note that $g_F(n_1,n_2,\ldots) = 0$ unless only finitely many of the $n_i$ are nonzero.

Let $D_i$ ($i \in I$) be the linear endomorphism that acts on $g_F(n_1,n_2,\ldots)$ as multiplication by the scalar $n_i$. Then $D_i$ is a derivation. Call it the $i^{th}$ degree derivation of $g_F$. Since the ideal $s$ is homogeneous under the above grading, $g_0$ can also be graded by $\mathbb{Z}$, and the $D_i$ can be transferred naturally to $g_0$. Let $\eta$ be the automorphism of order 2 of $g_F$ taking $h_i$ to $-h_i$ and interchanging $e_i$ and $f_i$ for all $i \in I$. Since $\eta(s) = s$, $\eta$ is well defined on $g_0$.

We use the same symbols $h_i,e_i$, and $f_i$ for their images in $g_0$. Let $\mathfrak{h}$ be the abelian subalgebra spanned by the $h_i$ for all $i \in I$. Let $g_0^+$ be the subalgebra generated by the $e_i$ for $i \in I$ and let $g_0^-$ be the subalgebra generated by the $f_i$ for $i \in I$. The space $g_0^- \oplus \mathfrak{h} \oplus g_0^+$ is a subalgebra of $g_0$, and since $g_0 \oplus \mathfrak{h} \oplus g_0^+$ contains the generators $e_i,f_i,h_i$, it follows that $g_0 = g_0^- \oplus \mathfrak{h} \oplus g_0^+$. If the $n_i$ are nonnegative (resp. nonpositive) and only finitely many are nonzero, then $g_0(n_1,n_2,\ldots)$ is spanned by the elements $[e_i,e_{i_2}\cdots e_{i_{r-1}},e_{i_r}]$ (respectively, $[f_i,f_{i_2}\cdots f_{i_{r-1}},f_{i_r}]$) where $e_j$ (or $f_j$) occurs $|n_j|$ times.

**Proposition 1.1.** The Lie algebra $g_0$ has triangular decomposition

$$g_0 = g_0^- \oplus \mathfrak{h} \oplus g_0^+.$$ 

The abelian subalgebra $\mathfrak{h}$ has basis consisting of the $h_i$, $i \in I$. Furthermore, $g_0^\pm$ is the free Lie algebra generated by the $e_i$ (respectively, the $f_i$) for $i \in I$. In particular, \{$e_i,f_i,h_i\}_{i \in I}$ is a linearly independent set in $g_0$.

**Proof.** As in the classical case, we will construct a sufficiently large representation of the Lie algebra. Let $\mathfrak{h}_F$ be the span of the $h_i$, $i \in I$. Define $\alpha_j \in (\mathfrak{h}_F)^*$ as follows:

$$\alpha_j(h_i) = a_{ij}.$$ 

Let $X$ be the free associative algebra on the set $\{x_i\}_{i \in I}$. Let $\lambda \in (\mathfrak{h}_F)^*$. A representation of $g_F$ on $X$ is given by the following action of the generators of $g_F$ on $X$:

$$h \cdot x_i = \lambda(h)x_i \quad \text{for all } h \in \mathfrak{h}_F$$

$$h \cdot x_{i_1}\cdots x_{i_r} = (\lambda - \alpha_{i_1} - \cdots - \alpha_{i_r})(h)x_{i_1}\cdots x_{i_r} \quad \text{for all } h \in \mathfrak{h}_F$$

$$f_i \cdot x_i = x_{i+1}$$

$$f_i \cdot x_{i_1}\cdots x_{i_r} = x_{i_1}x_{i_2}\cdots x_{i_r}$$

$$e_i \cdot x_i = 0$$

$$e_i \cdot x_{i_1}\cdots x_{i_r} = x_{i_1}e_i x_{i_2}\cdots x_{i_r} + \delta_{i_1}(\lambda - \alpha_{i_2} - \cdots - \alpha_{i_r})(h)x_{i_2}x_{i_3}\cdots x_{i_r}.$$ 

To show that $X$ can be regarded as a $g_0$-module we verify that the ideal $s$ annihilates the module $X$. It is clear that $[h,h'] = 0$ on $X$ for $h,h' \in \mathfrak{h}$. The element $[e_i,f_j] =$ —
\[ \delta_{ij} h_i \text{ also acts as } 0 \text{ on } X \text{ by the following computation :} \]
\[
\begin{align*}
[e_i, f_j] \cdot x_{i_1} \cdots x_{i_r} &= e_i f_j x_{i_1} \cdots x_{i_r} - f_j e_i x_{i_1} \cdots x_{i_r} \\
 &= e_i \cdot x_j x_{i_1} \cdots x_{i_r} - x_j e_i \cdot x_{i_1} \cdots x_{i_r} \\
 &= \delta_{ij} (\lambda - \alpha_{i_1} - \cdots - \alpha_{i_r})(h_i) x_{i_1} \cdots x_{i_r} \\
 &= \delta_{ij} h_i \cdot x_{i_1} \cdots x_{i_r}.
\end{align*}
\]

By a similar computation \([h_i, e_k] + a_{ik} f_k\) annihilates \(X\). Now consider the action of \([h_i, e_k] - a_{ik} e_k\) on \(X\):
\[
[h_i, e_k] \cdot 1 = h_i e_k \cdot 1 - e_k h_i \cdot 1
\]
and
\[
a_{ik} e_k \cdot 1 = 0.
\]
Thus \([h_i, e_k] - a_{ik} e_k\) annihilates \(1\). Furthermore, using the above results
\[
\begin{align*}
[[h_i, e_k] - a_{ik} e_k, f_l] &= [[h_i, e_k], f_l] - [a_{ik} e_k, f_l] \\
&= [e_k [f_l, h_i]] + [h_i [e_k, f_l]] - [a_{ik} e_k, f_l] \\
&= 0.
\end{align*}
\]
Thus \([h_i, e_k] - a_{ik} e_k\) commutes with the action of \(f_l\) for all \(l\). Since any \(x_{i_1} \cdots x_{i_r} = f_{i_1} \cdots f_{i_r} \cdot 1\), this means that \([h_i, e_k] - a_{ik} e_k\) annihilates \(X\). Thus \(X\) can be regarded as an \(g_0\)-module. That the \(h_i\)'s are linearly independent follows from the first relation, since \(\lambda \in \mathfrak{h}_F\) is allowed to vary.

From the definition of the module we see that \(\mathcal{U}(g^-)\) generates the regular representation of \(X\), so there is a homomorphism (of associative algebras):
\[
\mathcal{U}(g^-) \rightarrow X
\]
\[
f_i \mapsto x_i.
\]
If \(g_X\) is the free Lie algebra generated by \(\{x_i\}_{i \in I}\) then there is an associative algebra homomorphism:
\[
\mathcal{U}(g_X) \rightarrow \mathcal{U}(g^-)
\]
\[
x_i \mapsto f_i.
\]
But \(\mathcal{U}(g_X) = X\), so the map \(\Phi\) is an isomorphism. Thus \(g_X\) is isomorphic to \(g^-\), the Lie subalgebra in \(\mathcal{U}(g^-) = \mathcal{U}(g_X)\) generated by the \(f_i, i \in I\). Applying \(\eta\) shows that \(g_+\) is the free Lie algebra generated by \(\{e_i\}_{i \in I}\). Thus the set of \(e_i\)'s (and \(f_i\)'s) are linearly independent, and since \(g_0 = g^- \oplus \mathfrak{h} \oplus g_+\) any finite collection of the \(h_i\), \(e_i\) and \(f_i\) for \(i \in I\) together is linearly independent. \(\square\)

**Definition.** Let \(u_i\) denote the subalgebra of \(g_0\) spanned by the elements \(h_i, e_i, f_i\).
Remark. If $a_{ii} \in \mathbb{E}$ is nonzero, then the Lie algebra $\mathfrak{u}_i$ is isomorphic to $\mathfrak{sl}_2$. This is easily seen by rescaling $h_i$ and $e_i$, so that

$$\tilde{h}_i = \frac{2h_i}{a_{ii}}, \tilde{e}_i = \frac{2e_i}{a_{ii}}.$$ 

If $a_{ii} = 0$ then $\mathfrak{u}_i$ is a Heisenberg algebra.

Assume that we have a distinguished subset $I_0 \subset I$ such that for all $i \in I_0$,

1. $a_{ii} \neq 0$
2. $2a_{ij}/a_{ii} \in -\mathbb{N}$ ($i \neq j \in I$)
3. $a_{ij} = 0$ implies $a_{ji} = 0$ for all $j \in I$.

For all $i \neq j$ and $i \in I_0$ define

$$d_{ij}^+ = (ad e_i)(ad f_i)x = [f_i, [e_k, x]] = (ad f_i)(ad e_k)x$$

so

$$[e_k, (ad f_i)^{-2a_{ij}}+1 f_i] = (ad f_i)^{-2a_{ij}}+1 [e_k, f_i] = 0$$

Now assume $k = i$. Consider the $\mathfrak{sl}_2$-module generated by the highest weight vector $f_j$, with the adjoint action. Let $n = 2a_{ij}/a_{ii}$,

$$\begin{align*}
(\text{ad } e_i)(\text{ad } f_i)^{1-n} f_j &= (\text{ad } f_i)(\text{ad } e_i) f_j + (\text{ad } h_i) f_j, \\
&= \begin{cases} 
(\text{ad } f_i)^{1-n}(\text{ad } e_i) f_j + (-n + 1)(\text{ad } f_i)^{-n}(\text{ad } h_i) f_j + \frac{1}{2}(-n + 1)(-n)(\text{ad } f_i)^{-n-1}(\text{ad } (-a_{ii}f_i)) f_j, & \text{if } n < 0 \\
(h_i, f_j) & \text{if } n = 0
\end{cases} \\
&= \begin{cases} 
(-n + 1)(-a_{ij})(\text{ad } f_i)^{-n} f_j + 2a_{ii}(-n + 1)(n)(\text{ad } f_i)^{-n} f_j & \text{if } n < 0.
\end{cases}
\end{align*}$$

By the definition $n = 0$ if and only if $a_{ij} = 0$, furthermore $\frac{2a_{ii}}{a_{ii}}(n) = a_{ij}$. Thus both of the above terms are zero, and $\text{ad } e_i$ annihilates $(\text{ad } f_i)^{-2a_{ij}}+1 f_j$.

Finally assume $k = j$. By equation (1.1),

$$[e_j, (\text{ad } f_i)^{-2a_{ij}}+1 f_j] = (\text{ad } f_i)^{-2a_{ij}}+1 h_j$$

Since $[f_i, h_j] = a_{ij} f_i$, it follows immediately that (1.2) is zero if $2a_{ij}/a_{ii} < 0$. If $a_{ij} = 0$ then $a_{ji} = 0$ and (1.2) is also zero. \qed
Let $\mathfrak{k}_0^\pm$ be the ideal of $\mathfrak{g}_0^\pm$, respectively, generated by the elements:

\[
d_{ij}^\pm \text{ for all } i \in I_0, i \neq j \in I
\]

\[
[e_i, e_j] \text{ if } a_{ij} = a_{ji} = 0 \text{ for } \mathfrak{k}_0^+
\]

\[
[f_i, f_j] \text{ if } a_{ij} = a_{ji} = 0 \text{ for } \mathfrak{k}_0^-
\]

(1.3)

Note that the last elements are not redundant if $i \in I \setminus I_0$.

Let $n_1 = \mathfrak{g}_0^+ / \mathfrak{k}_0^+$ and $n^{-1}_1 = \mathfrak{g}_0^- / \mathfrak{k}_0^-$. Define $\mathfrak{k}_0 = \mathfrak{k}_0^+ \oplus \mathfrak{k}_0^-$. 

**Proposition 1.3.** The subalgebras $\mathfrak{k}_0^\pm$ and $\mathfrak{k}_0$ are ideals of $\mathfrak{g}_0$.

**Proof.** Let $\mathfrak{g}_0$ act on itself by the adjoint representation. To see that $\mathfrak{k}_0^-$ is an ideal, first consider the action on the generators $d_{ij}^-$. By Proposition 1.1:

\[
\sum_{i \neq j} \mathcal{U}(\mathfrak{g}_0) \cdot d_{ij}^- = \sum_{i \neq j} \mathcal{U}(\mathfrak{g}_0^-) \mathcal{U}(\mathfrak{h}) \mathcal{U}(\mathfrak{g}_0) \cdot d_{ij}^-
\]

\[
= \sum_{i \neq j} \mathcal{U}(\mathfrak{g}_0^-) \cdot d_{ij}^- \subset \mathfrak{k}_0^-
\]

The last equality holds by Proposition 1.2 and the fact that for $h \in \mathfrak{h}$,

\[
[h, (\text{ad } f_i)^N f_j] = \lambda (\text{ad } f_i)^N f_j
\]

where $\lambda$ is a scalar.

We must also consider the generators of the form (1.3). By the Jacobi identity, and the assumption that $a_{ij} = a_{ji} = 0$:

\[
[e_k [f_i, f_j]] = [\delta_{ki} h_i, f_j] + [f_i, \delta_{kj} h_j]
\]

\[
= -\delta_{ki} a_{ji} f_j + \delta_{kj} a_{ji} f_i
\]

\[
= 0.
\]

Thus we can use the same argument appearing above for $d_{ij}^-$ to conclude

\[
\sum_{i \neq j, a_{ij} = a_{ji} = 0} \mathcal{U}(\mathfrak{g}_0) \cdot [f_i, f_j] \subset \mathfrak{k}_0^-
\]

By a symmetric argument, $\mathfrak{k}_0^+$ is an ideal of $\mathfrak{g}_0$, therefore $\mathfrak{k}_0 = \mathfrak{k}_0^+ \oplus \mathfrak{k}_0^-$ is also an ideal. 

Define the Lie algebra $\mathfrak{g}_1 = \mathfrak{g}_0 / \mathfrak{k}_0$. The Lie algebra $\mathfrak{g}_1$ is given by the corresponding generators and relations. Since the ideal $\mathfrak{k}_0$ is homogeneous, $\mathfrak{g}_1$ has a $\mathbb{Z}^I$-gradation induced by the given grading of $\mathfrak{g}_0$, and $\eta$ is well defined on $\mathfrak{g}_1$. We use the same letters $e_i, f_i, h_i$ and $\mathfrak{h}$ to denote their images in $\mathfrak{g}_1$. The Lie algebra $\mathfrak{g}_1$ has a unique graded ideal, $\mathfrak{r}_1$, maximal among graded ideals not intersecting $\mathfrak{h}$. The ideal $\mathfrak{r}_1$ does not intersect the span of the $h_i, e_i$, and $f_i$s. The fact that $\eta(\mathfrak{r}_1) \subset \mathfrak{r}_1$ is easy to verify.
Definition. The Lie algebra \( g = g(A) \) is defined to be \( g_1 / r_1 \).

We use the same letters \( h_i, e_i, f_i, u_i \) and \( h \) for their images in \( g \). The map \( \eta \) and the \( D_i \)'s are well defined on \( g \). It is still true that if the \( n_i \)'s are all nonnegative (resp. nonpositive), and only finitely many of the \( n_i \) are nonzero, then \( g(n_1, n_2 \ldots) \) is spanned by the elements:

\[
[e_{i_1} e_{i_2} \cdots [e_{i_{r-1}}, e_{i_r}] \cdots]
\]

(resp. \( [f_{i_1} [f_{i_2} \cdots [f_{i_{r-1}}, f_{i_r}] \cdots] \))

where \( e_j \) (resp. \( f_j \)) occurs \( |n_j| \) times. The space \( g(0, 0, \pm 1, 0, \ldots) \) where \( \pm 1 \) occurs in the \( i^{th} \) position is spanned by \( e_i \) (resp. \( f_i \)).

Define the subalgebras

\[
n^+ = n = n_1 / (n_1 \cap r_1) \quad \text{and} \quad n^- = n_1^- / (n_1^- \cap r_1).
\]

Then, since \( g_0 = g^- \oplus h \oplus g^+ \), we have the triangular decomposition \( g_1 = n^- \oplus h \oplus n_1 \), and therefore \( g = n^- \oplus h \oplus n \), using the same notation for \( h \) in the Lie algebras \( g_0, g_1 \) and \( g \).

Proposition 1.4. For each \( i \in I_0 \), \( g \) is a direct sum of finite-dimensional irreducible \( u_i \)-modules.

Proof. Since \( u_i \) is isomorphic to \( \mathfrak{sl}_2 \) the usual proof for Kac-Moody algebras works in this case. \( \square \)

1.2 Definition of a generalized Kac-Moody algebra.

First we construct the extended Lie algebra \( g^e \). Let \( d_0 \) denote the (possibly infinite-dimensional) space of commuting derivations of \( g \) spanned by the \( D_i \) for \( i \in I \). Let \( d \) be a subspace of \( d_0 \). Form the semidirect product Lie algebra \( g^e = d \rtimes g \). Then \( h^e = d \rtimes h \) is an abelian subalgebra of \( g^e \) which acts via scalar multiplication on each space \( g(n_1, n_2, \ldots) \).

Let \( \alpha_i \in (h^e)^\ast, i \in I \), be defined by the conditions:

\[
[h, e_i] = \alpha_i(h)e_i \quad \text{for all} \quad h \in h^e.
\]

Note that \( \alpha_j(h_i) = \alpha_{ij} \) for all \( i, j \in I \). Choose \( d \) so that the \( \alpha_i \) for \( i \in I \) are linearly independent. This is possible because this condition holds for \( d = d_0 \). If \( t \) is a \( d \)-invariant subalgebra of \( g \), denote by \( t^e \) the subalgebra \( d \rtimes t \) of \( g^e \).

For all \( \varphi \in (h^e)^\ast \) define

\[
g^\varphi = \{ x \in g | [h, x] = \varphi(h)x \text{ for all } h \in h^e \}.
\]

If \( \varphi, \psi \in (h^e)^\ast \) then \( [g^\varphi, g^\psi] \subset g^{\varphi + \psi} \). By definition \( e_i \in g^{\alpha_i}, f_i \in g^{-\alpha_i} \) for all \( i \in I \), so that if all \( n_i \leq 0 \) or all \( n_i \geq 0 \) (only finitely many nonzero) it is clear that

\[
g(n_1, n_2, \ldots) \subset g^{n_1 \alpha_1 + n_2 \alpha_2 + \cdots}.
\]

In fact, since the \( \alpha_i \) for \( i \in I \) are linearly independent

\[
g^{n_1 \alpha_1 + n_2 \alpha_2 + \cdots} = g(n_1, n_2, \ldots)
\]

and \( g^0 = h \). Therefore,
(1.4) \[ g = \bigoplus_{(n_1, n_2, \ldots) \in \mathbb{Z}^I} g^{n_1 \alpha_1 + n_2 \alpha_2 + \cdots}. \]

**Remark.** The center of \( g \) is contained in \( h \). We observe that \( g^0 = 0 \) unless \( n = \pm 1 \).

**Definition.** The roots of \( g \) are the nonzero elements \( \varphi \) of \((h^\vee)^*\) such that \( g^\varphi \neq 0 \). The elements \( \alpha_i \) are the **simple** roots. Let \( \Delta \) be the set of roots, \( \Delta_+ \) the set of roots which are non-negative integral linear combinations of \( \alpha_i \)’s. The roots in \( \Delta_+ \) are called the **positive** roots. Let \( \Delta_- = -\Delta_+ \) be the set of **negative** roots. All of the roots are either positive or negative.

The elementary properties of \( g \) are summarized in the following proposition.

**Proposition 1.5.** The Lie algebra \( g \) has the following properties:
1. \( g = n^- \oplus h \oplus n^+ \)
2. \( h = g^0 \) and \( h \) has a basis \( h_i, i \in I \)
3. \( n^+ = \bigoplus_{\varphi \in \Delta_+} g^\varphi \) and \( n^- = \bigoplus_{\varphi \in \Delta_-} g^\varphi \)
4. \( \Delta = \Delta_+ \cup \Delta_- \) (disjoint union)
5. \( [g^\varphi, g^\psi] \subset g^{\varphi + \psi} \) for all \( \varphi, \psi \in (h^\vee)^* \)
6. \( \eta(g^\varphi) = g^{-\varphi} \) for all \( \varphi \in (h^\vee)^* \), and \( \dim g^\varphi = \dim g^{-\varphi} < \infty \)
7. \( g^{\pm \alpha_i} \) has basis \( \{e_i\} \) resp., \( \{f_i\} \) for all \( i \in I \)
8. For each \( i \in I_0 \) every \( \text{ad} u_i \)-stable subspace of \( g \) is a direct sum of finite-dimensional \( u_i \)-modules.
9. \( \alpha_j(h_i) = a_{ij} \) for all \( i, j \in I \)
10. Every nonzero ideal of \( g^\varphi \) meets \( h^\varphi \).
11. If the (possibly infinite) matrix \( A' \) is obtained from the matrix \( A \) by a permutation of the rows and a corresponding permutation of columns, then there is a natural isomorphism \( g(A) \cong g(A') \).
12. If \( A \) has the form \( \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \), there is a natural isomorphism \( g(A) \cong g(B) \times g(C) \).

For the definition of generalized Kac-Moody algebra we assume that \( A = (a_{ij})_{i,j \in I} \) is a matrix with entries in \( \mathbb{F} \), satisfying the following conditions:

(C1) \( A \) is symmetrizable.
(C2) If \( a_{ii} > 0 \), then \( a_{ii} = 2 \). For all \( k \neq j \), \( a_{jk} \leq 0 \), and for all \( i \in I_0 \), \( a_{ij} \in -\mathbb{N} \) \((j, k \in I)\).
(C3) If \( a_{ij} = 0 \), then \( a_{ji} = 0 \) \((i, j \in I)\).

(Recall a matrix \( A \) is called **symmetrizable** if there are positive nonzero numbers \( q_i \in \mathbb{F}, i \in I \), such that \( DA \) is symmetric, where \( D \) is the diagonal matrix with entries \( q_i \). Note that this condition implies (C3).)

We also assume that \( I_0 \) is chosen to be as large as possible, so that \( I_0 = \{i \in I \mid a_{ii} > 0\} \).

**Definition.** The **generalized Kac-Moody (Lie) algebra** associated to the matrix \( A \) is \( g = g(A) \).
1.3 An invariant bilinear form on \( \mathfrak{g}^e \).

**Definition.** We define a symmetric bilinear form \((\cdot,\cdot)\) taking values in \( \mathbb{E} \) on \( R \) by defining \((\alpha_i,\alpha_j) = q_{ij}a_{ij}\) and extending linearly to any element of \( \Delta \) (or \( R \)). Notice that on \( \Delta \) the form \((\cdot,\cdot)\) actually takes values in \( \mathbb{F} \).

For all \( i \in I \), let \( x_{\alpha_i} = q_i h_i \in \mathfrak{h} \). For \( \varphi \in R \) with \( \varphi = \sum a_i \alpha_i \), define \( x_{\varphi} = \sum a_i x_{\alpha_i} \in \mathfrak{h} \). We can extend the \( \mathbb{E} \)-valued symmetric form \((\cdot,\cdot)\) defined on \( R \) to a symmetric form on \((\mathfrak{h}^e)^*\) satisfying the condition:

\[(\lambda,\alpha_i) = \lambda(x_{\alpha_i}) \quad \text{for all } \alpha_i \text{ and } \lambda \in (\mathfrak{h}^e)^*.\]

Fix such a form.

The map given by \( \varphi \mapsto x_{\varphi} \) defines a linear isomorphism from \( R \) to \( \mathfrak{h} \). The bilinear form on \( R \) can be transferred to a form \((\cdot,\cdot)_{\mathfrak{h}}\) on \( \mathfrak{h} \) by letting \((x_{\alpha_i},x_{\alpha_j})_{\mathfrak{h}} = (\alpha_i,\alpha_j) = q_{ij}a_{ij}\) for all \( i,j \in I \). Note that \((x_{\psi},x_{\varphi})_{\mathfrak{h}} = (\psi,\varphi) = \psi(x_{\varphi}) = \varphi(x_{\psi})\), since this is true for the \( x_{\alpha_i}, i \in I \).

To extend the above form to a symmetric bilinear form on \( \mathfrak{h}^e = \mathfrak{d} \oplus \mathfrak{h} \), let \((D_i,x_{\alpha_j})_{\mathfrak{h}^e} = \alpha_j(D_i) = \delta_{ij}\) for all \( i,j \in I \), and pick an arbitrary symmetric bilinear form on \( \mathfrak{d} \). This determines a form \((\cdot,\cdot)_{\mathfrak{h}^e}\) on \( \mathfrak{h}^e \) that satisfies

\[(1.5) \quad (h,x_{\varphi})_{\mathfrak{h}^e} = \varphi(h)\]

for all \( h \in \mathfrak{h}^e \) and \( \varphi \in R \).

**Remark.** Note that \( x \in \mathfrak{h} \) is determined by the values \((h,x)_{\mathfrak{h}^e}\) as \( h \) ranges through \( \mathfrak{h}^e \).

Recall that a symmetric bilinear form \((\cdot,\cdot)\) on a Lie algebra \( \mathfrak{g} \) is called *invariant* if

\[(a,b,c) = -([a,[b,c]])\]

for all \( a,b,c \in \mathfrak{g} \).

The bilinear form on \( \mathfrak{h}^e \) will be extended to a unique \( \mathfrak{g}^e \)-invariant bilinear form by using induction.

**Theorem 1.6.** There is a unique invariant symmetric bilinear form \((\cdot,\cdot)\) on \( \mathfrak{g}^e \) which extends \((\cdot,\cdot)_{\mathfrak{h}^e}\). For all \( \varphi,\psi \in \Delta \) such that \( \varphi + \psi \neq 0 \),

\[(\mathfrak{g}^e,\mathfrak{g}^e) = (\mathfrak{h}^e,\mathfrak{g}^e) = 0\]

and

\[[a,b] = (a,b)_{\mathfrak{h}^e}\]

for all \( a \in \mathfrak{g}^e \) and \( b \in \mathfrak{g}^{-\mathfrak{e}} \). In particular, \([\mathfrak{g}^e,\mathfrak{g}^{-\mathfrak{e}}] \subset \mathfrak{E}x_{\varphi} \). Moreover, \((e_i,f_j) = \frac{1}{\delta_{ij}}\delta_{ij}\) for all \( i,j \in I \).

**Proposition 1.7.** A form \((\cdot,\cdot)\) on \( \mathfrak{g}^e \) that extends \((\cdot,\cdot)_{\mathfrak{h}^e}\) is \( \mathfrak{h}^e \)-invariant if and only if for all \( \varphi,\psi \in \Delta \) such that \( \varphi + \psi \neq 0 \), we have

\[(\mathfrak{g}^e,\mathfrak{g}^e) = (\mathfrak{h}^e,\mathfrak{g}^e) = 0.\]

**Proof.** Given \( \alpha,\beta \in \Delta \cup \{0\}, h \in \mathfrak{h}^e \), and \( a \in \mathfrak{g}^e, b \in \mathfrak{g}^\beta \). If \((\cdot,\cdot)\) is \( \mathfrak{h}^e \)-invariant, then \((h,a),b) = -(a,[h,b])\). Therefore, \((\alpha+\beta)(h)(a,b) = 0\). If \( \alpha + \beta \neq 0 \), choose \( h \) such that \((\alpha+\beta)(h) \neq 0 \) so that \((a,b) = 0\) \( \square \)
Proposition 1.8. If \( g^e \) has an invariant bilinear form \((\cdot,\cdot)\) which extends \((\cdot,\cdot)_h\), then it is unique. For all \( \varphi \in \Delta, a \in g^e, \) and \( b \in g^{-\varphi}, \)
\[
[a, b] = (a, b)x_\varphi.
\]
In particular, the existence of \((\cdot,\cdot)\) implies that \([g^e, g^{-\varphi}] \subset \mathbb{E}x_\varphi.\)

Proof. By Proposition 1.7 it is sufficient to show that \([a, b] = (a, b)x_\varphi\). Let \( h \in h^e \). Then
\[
(h, [a, b])_{h^e} = (h, [a, b]) = ([h, a], b) = \varphi(h)(a, b) = (h, x_\varphi)_{h^e}(a, b) = (h, (a, b)x_\varphi)_{h^e}.
\]
Thus \((a, b) - (a, b)x_\varphi\) corresponds to an element of \( R^* \) which vanishes on all \( h \in h^e \), so we have \([a, b] = (a, b)x_\varphi\).

Consider the following grading of \( g^e \): For \( \varphi \in R \) such that \( \varphi = \sum_{i \in I} n_i \alpha_i \) where \( n_i \in \mathbb{Z} \), define the height \( ht(\varphi) \) to be the integer \( \sum_{i \in I} n_i \). If \( \varphi \in \Delta_+ \) then \( ht(\varphi) > 0 \), and if \( \varphi \in \Delta_- \) then \( ht(\varphi) < 0 \). For all \( n \in \mathbb{Z}_+ \), define
\[
\Delta(n) = \{ \varphi \in \Delta \mid \text{ht}(\varphi) \leq n \}.
\]
The subspace \( g^e(n) \) of \( g^e \) is defined by:
\[
g^e(n) = h^e \oplus \coprod_{\varphi \in \Delta(n)} g^\varphi.
\]
A symmetric bilinear form \((\cdot,\cdot)\) on \( g^e(n) \) is called invariant if for all \( \varphi_1, \varphi_2, \varphi_3 \in \Delta(n) \cup \{0\} \) such that \( |ht(\varphi_1 + \varphi_2)| \leq n \) and \( |ht(\varphi_1 + \varphi_3)| \leq n \), we have
\[
([a_1, a_2], a_3) = -(a_2, [a_1, a_3])
\]
for all \( a_i \in g^\varphi \) or in \( h^e \) if \( \varphi_i = 0 \).

The invariant form will now be defined inductively on \( g^e(n) \). Define \((\cdot,\cdot)_1\) on \( g^e(1) \) by letting \((\cdot,\cdot)_1 = (\cdot,\cdot)_{h^e}\) on \( h^e \), and by the conditions
\[
(e_i, e_j)_1 = (f_i, f_j)_1 = 0 \quad \text{for all } i, j \in I,
\]
\[
(h, g^\varphi) = 0 \quad \text{for all } h \in h^e, \varphi \in \Delta(1),
\]
and
\[
(e_i, f_j)_1 = \delta_{ij} \frac{1}{q_i} \quad \text{for all } i, j \in I.
\]
Then \((\cdot,\cdot)_1\) is invariant, and clearly satisfies \([a, b] = (a, b)_1 x_\varphi\) for all \( \varphi \in \Delta(1),\ a \in g^e b \in g^{-\varphi}.\)

Now assume that \( n > 0 \) and that \((\cdot,\cdot)_n\) is an invariant form on \( g^e(n) \) which extends \((\cdot,\cdot)_{h^e}\). We will extend the form \((\cdot,\cdot)_n\) to an invariant form on the space \( g^e(n + 1).\)
LEMMA 1.9. For all $\varphi, \psi \in \Delta(n)$, $(g^\varphi, g^\psi)_n = 0$ unless $\varphi + \psi = 0$ and $(h^\varphi, g^\varphi)_n = 0$. Furthermore,

$$[a, b] = (a, b)_n x_{\varphi} \text{ for all } a \in g^\varphi, b \in g^{-\varphi}.$$  

PROOF. See the proofs of Propositions 1.7 and Proposition 1.8. \hfill \Box

LEMMA 1.10. For all $\varphi \in \Delta(n + 1)$, $[g^\varphi, g^{-\varphi}] \subset Ex_{\varphi}$.

PROOF. By Lemma 1.9, we may assume $\varphi \in (\Delta(n + 1) \setminus \Delta(n)) \cap \Delta_+$. Let $a \in g^\varphi$ and $b \in g^{-\varphi}$. Without loss of generality we may assume, for some $i, j \in I$,

$$a = [e_i, a']$$

$$b = [f_j, b']$$

where $a' \in g^{\varphi - \alpha_i}$ and $b' \in g^{-\varphi + \alpha_i}$ (recall that $g^\varphi$ is spanned by elements of the form $[e_i, [e_{i_1}, \ldots, e_{i_{m-1}}, e_{i_m}]]$ with $\sum \alpha_{i_m} = \varphi$).

Then

$$[a, b] = [[e_i, a'], [f_j], b'] + [f_j, [[e_i, a'], b']]$$

$$= [[e_i, f_j], a'], b'] + [[e_i, [a, f_j]], b'] + [f_j, [[e_i, b'], a']] + [f_j, [e_j, [a', b']]]$$

$$= \delta_{ij}(\varphi - \alpha_i)(h_i)[a', b'] + ([e_i, [a', f_j]], b')a_{\varphi-\alpha_i}$$

$$+ (f_j, [[e_i, b'], a'])n x_{\varphi-\alpha_i} + \delta_{ij} \alpha_i([a', b'])h_i$$

$$= \delta_{ij}(a', b')n \{ (\varphi - \alpha_i)(h_i)x_{\varphi-\alpha_i} + \alpha_i(x_{\varphi-\alpha_i})h_i \}$$

$$+ ([e_i, [a', f_j]], b')n x_{\varphi} + ([e_i, [a', f_j]], b')n + (f_j, [[e_i, b'], a'])n x_{\varphi-\alpha_i}.$$  

Considering the first term, we see

$$(\varphi - \alpha_i)(h_i)x_{\varphi} + (h_i, [x_{\varphi-\alpha_i}, x_{\varphi-\alpha_i}])h_i$$

$$= (\varphi - \alpha_i)(h_i)x_{\varphi},$$

and by the invariance of $(\cdot, \cdot)_n$,

$$([e_i, [a', f_j]], b')n + (f_j, [[e_i, b'], a'])n = 0.$$  

\hfill \Box

PROOF OF THEOREM 1.6. By Lemmas 1.9 and 1.10, we can define $(\cdot, \cdot)_{n+1}$ to be the symmetric form on $g^\varphi (n + 1)$ which extends $(\cdot, \cdot)_n$, such that $[a, b] = (a, b)_{n+1} x_{\varphi}$ for all $a \in g^\varphi$, $b \in g^{-\varphi}$ with $\varphi \in \Delta(n + 1)$, and such that for $\varphi, \psi \in \Delta(n + 1)$,

$$(g^\varphi, g^\psi)_{n+1} = 0$$

unless $\varphi + \psi = 0$, and $(h^\varphi, g^\varphi) = 0$.

All that remains to be shown in order to prove Theorem 1.6 is that this form $(\cdot, \cdot)_{n+1}$ is invariant. Since no confusion will arise, we now write $(\cdot, \cdot)_{n+1} = (\cdot, \cdot)$.

Let $\varphi_1, \varphi_2, \varphi_3 \in \Delta(n + 1) \cup \{0\}$ such that $\text{ht} (\varphi_1 + \varphi_2) \leq n + 1$ and $\text{ht} (\varphi_1 + \varphi_3) \leq n + 1$.

We claim that if $a_i \in g^{\varphi_i}$ or $b^\varphi$ (in the case $\varphi_i = 0$), for $i = 1, 2, 3$, then

$$(a_1, a_2, a_3) = -(a_2, [a_1, a_3]).$$  

(1.6)
To prove this claim, we may assume that $\varphi_1 + \varphi_2 + \varphi_3 = 0$, and by the proofs of Propositions 1.7 and 1.8 that each $\varphi_i \neq 0$.

First suppose $\varphi_1$ and $\varphi_2$ are linearly independent. Let $a^1 = [a_2, a_3]$, $a^{-2} = [a_1, a_3]$ and $a^{-3} = [a_1, a_2]$, so that $a^i \in g^{-\varphi_i}$ ($i = 1, 2, 3$). Then

$$[a_1, a^1] = [a^{-3}, a_3] + [a_2, a^{-2}],$$

so that,

$$(1.7) \quad (a_1, a^1)x_{\varphi_1} = (a^{-3}, a_3)x_{-\varphi_3} + (a_2, a^{-2})x_{\varphi_2}.$$ 

But $x_{-\varphi_3} = x_{\varphi_1} + x_{\varphi_2}$ by the assumption $\varphi_1 + \varphi_2 + \varphi_3 = 0$ and so (1.7) gives

$$((a_1, a^1) - (a^{-3}, a_3))x_{\varphi_1} = ((a_2, a^{-2}) + (a^{-3}, a_3))x_{\varphi_2}.$$ 

By the definition of $x_\varphi$ we have

$$((a_1, a^1) - (a^{-3}, a_3))\varphi_1 = ((a_2, a^{-2}) + (a^{-3}, a_3))\varphi_2.$$ 

Thus linear independence of $\varphi_1$ and $\varphi_2$ gives:

$$(a_2, a^{-2}) + (a^{-3}, a_3) = 0$$

which is equivalent to equation (1.6).

Now suppose $\varphi_1$ and $\varphi_2$ are linearly dependent. Then for each $i = 1, 2, 3$ and $j \in I$, $\varphi_j \notin \mathbb{E} \alpha_j$. Assume that $\varphi_{i_0}$ is positive for $i_0$ one of 1, 2, 3 and is negative otherwise; this is enough because the automorphism $\eta$ can be applied. We may also assume that $\text{ht}(\varphi_{i_0}) = n + 1$, so that $|\text{ht}(\varphi_i)| \leq n$ for $i \neq i_0$. Since the roles of $\alpha_2$ and $\varphi_3$ are symmetric it is enough to show the claim for $i_0 = 1$ or 2. We may also assume that $a_{i_0} = [e_k, a_{i_0}']$ for some $k \in I$ and some $a_{i_0}' \in g^{\varphi_{i_0} - \alpha_k}$.

First let $i_0 = 1$. Then

$$(a_1, a_2), a_3) + (a_2, [a_1, a_3]) = ([e_k, a_1'], a_2), a_3) + (a_2, [e_k, a_1'], a_3))$$

$$= ([e_k, a_2], a_1'), a_3) + ([e_k, a_1]', a_2), a_3))$$

$$+ (a_2, [e_k, a_3], a_1') + (a_2, [e_k, a_1], a_3]).$$

By the invariance of $(\cdot, \cdot)_n$ this expression is zero.

Now suppose that $i_0 = 2$. Then

$$(1.8) \quad ([a_1, a_2], a_3) + (a_2, [a_1, a_3]) = ([e_k, a_2'], a_3) + ([e_k, a_2], [a_1, a_3]).$$

Because $\alpha_k$ and $\varphi_2 - \alpha_k$ are linearly independent, the previous argument applies to the second term on the right. Using invariance of $(\cdot, \cdot)_n$

$$([e_k, a_2], [a_1, a_3]) = -(a_2', [e_k, [a_1, a_3]])$$

$$= -(a_2', [e_k, a_1], a_3) - (a_2', [a_1, [e_k, a_3]])$$

$$= -([a_2', [e_k, a_1]], a_3) - ([a_2', a_1], [e_k, a_3])$$

$$= -([a_2', e_k], [a_1], a_3) - ([e_k, [a_2', a_1]], a_3) - ([a_2', a_1], [e_k, a_3])$$

$$= -([a_2', e_k], [a_1], a_3).$$

Thus equation (1.8) equals zero and we are done. \qed
1.4 The radical of $\mathfrak{g}_1$ is zero.

The theorem below follows from Theorem 1 in [GK] or Proposition 9.11 in [K], and a lemma in [B1]. The only generalization of [K] being that $\mathfrak{g}_0$ has a countable number of generators. This theorem is also proven in [HMY] in the same manner.

The Lie algebra $\mathfrak{g}_0$ defined in §1.1, which is denoted $\mathfrak{g}(A)$ in [K], can be graded as in [K] by the lattice $Q = \sum_{i \in I} \mathbb{Z} \alpha_i$ (agreeing with the $\mathbb{Z}^I$-grading in §1.1), and $\mathfrak{g}_1$ has an induced $Q$-grading, we will call this the root grading of $\mathfrak{g}_1$. Denote by $\Pi$ the set of simple roots of $\mathfrak{g}$ as in [B1] and a lemma in [B1].

Thus $\pi$ is zero because we can assume $\pi = 0$.

Let $\eta_0$ be the radical of $\mathfrak{g}_0$. Then note that if $\pi$ is the natural map $\pi : \mathfrak{g}_0 \rightarrow \mathfrak{g}_1$ then $\pi(\eta_0) = \eta_1$. Let $\eta_0^+ = \eta_0 \cap \mathfrak{n}_1^+$, also denote $(\eta_i)^\alpha_i = \eta_i \cap \mathfrak{g}_\alpha_i$ for $\alpha \in Q, i = 0, 1$. Let $Q_+ = \sum_{i \in I} \mathbb{N} \alpha_i$.

**Theorem 1.11.** The ideal $\eta_1 \subset \mathfrak{g}_1$ is zero.

We restate the following result as it appears in [K], where $\rho \in \mathfrak{h}^*$ satisfies $2 = 1/2(\alpha_i, \alpha_i)$:

**Lemma 1.12.** The ideal $\eta_0^+$ (resp. $\eta_0^-$) is generated as an ideal in $\mathfrak{n}_1^+$ (resp. $\mathfrak{n}_1^-$) by those $(\eta_0)^\alpha_i$ (resp. $(\eta_0)^{-\alpha_i}$) for which $\alpha \in Q_+ \cap \Pi$ and $2(\rho, \alpha) = (\alpha, \alpha)$.

**Lemma 1.13.** Let $\alpha = \sum_{i \in I} k_i \alpha_i$, where $k_i \in \mathbb{N}$, be such that $(\mathfrak{g}_1)^\alpha \neq 0$. Then $\text{supp}(\alpha)$ cannot be written as a disjoint union, $S_1 \cup S_2$, of two proper subsets such that $(\alpha_i, \alpha_j) = 0$ for all $i \in S_1, j \in S_2$.

**Proof.** Given $\alpha = \sum_{i \in I} k_i \alpha_i$ as in the lemma, we know that $\mathfrak{g}_1^\alpha$ is spanned by elements of the form $[e_{i_1}, [e_{i_2}, \ldots [e_{i_{n-1}}, e_{i_n}] \ldots]$. The lemma is obvious if $\text{ht}(\alpha) < 2$. If $\text{ht}(\alpha) \geq 2$ the lemma is proven by induction on $\text{ht}(\alpha)$. Suppose $\text{ht}(\alpha) = 2$, and that $\text{supp}(\alpha) = S_1 \cup S_2$ where $S_1$ and $S_2$ are as in the lemma. Any $[e_{i_1}, e_{i_2}] \in \mathfrak{g}^\alpha$ is zero because we can assume $i_1 \in S_1$ and $i_2 \in S_2$ so that $\langle \alpha_{i_1}, \alpha_{i_2} \rangle = 0$, $a_{i_1} a_{i_2} = 0$ thus $\alpha$ is not a root. If the lemma is true for $\text{ht}(\alpha) = n - 1$, then $\text{ht}(\alpha) = n$ follows from the Jacobi identity.

The following lemma appears in [B1].

**Lemma 1.14.** Let $\alpha$ be an element of $Q_+ \setminus \{0\}$ such that $(\alpha, \alpha_i) \leq 0$ for all $i \in I_0$. Then $(\alpha, \alpha) - 2(\alpha, \rho) \leq 0$ and equality holds if and only if $\alpha = \alpha_i$ where $(\alpha, \alpha_i) = 0$.

**Proof.** Let $\alpha = \sum_{i \in I} k_i \alpha_i, \ k_i \geq 0$. Then

$$ (\alpha, \alpha) - 2(\alpha, \rho) = \sum_{i \in I} k_i (\alpha, \alpha - \alpha_i). $$

If $\alpha_i$ is such that $(\alpha_i, \alpha_i) \leq 0$, since $a_{ij} \leq 0$ for $i \neq j$ it follows that $k_i (\alpha_i, \alpha - \alpha_i) \leq 0$.

If $\alpha_i$ is such that $k_i \neq 0$ and $(\alpha_i, \alpha_i) > 0$, then $-k_i (\alpha_i, \alpha_i) < 0$ and since $(\alpha, \alpha_i) \leq 0$ we conclude $k_i (\alpha, \alpha_i) \leq 0$. Thus equation (1.9) implies the desired inequality.

For equality to hold all the $\alpha_i$ must satisfy $(\alpha_i, \alpha_i) \leq 0$ and $(\alpha_i, \alpha - \alpha_i) = 0$. Thus

$$ \sum_{i \neq j} k_j (\alpha_i, \alpha_j) + (k_i - 1)(\alpha_i, \alpha_i) = 0. $$

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Since all of the above summands are nonpositive, we conclude that \((\alpha_i, \alpha_j) = 0\) if \(k_j \neq 0\), and if \(k_i > 1\) then \((\alpha_i, \alpha_i) = 0\). Thus by Lemma 1.13, equality holds if and only if \(\alpha = \alpha_i\) where \((\alpha_i, \alpha_i) = 0\). \(\square\)

**Proof of Theorem 1.11.** (See the proof of Theorem 2 in [GK] or Theorem 9.11 in [K]). Let \(r_1 = \prod_{\alpha \in Q} r^\alpha_1\) be the grading induced by the \(Q\)-grading on \(g_1\).

If \(i \in I_0\) there exists automorphisms of \(Q\) given by

\[ r_i(\alpha_j) = \alpha_j - a_{ij} \alpha_i \]

and for each \(r_i\) there exists \(\tilde{r}_i \in \text{Aut}(g_1)\) such that

\[ \tilde{r}_i(g^\alpha_1) = g^{r^\alpha_1}_1. \]

Assume that \(r^+_1 \neq 0\), then we can choose \(\alpha = \sum_{i \in I} k_i \alpha_i \in Q_+ \setminus \{0\}\) such that \(r^+_1 \neq 0\) and \(ht(\alpha)\) is minimal. Thus \(ht(r_i(\alpha)) \geq ht(\alpha)\) for all \(i \in I_0\). This means that \((\alpha, \alpha_i) \leq 0\) for all \(i \in I_0\). Hence we can apply Lemma 1.14. Now consider \((\alpha, \alpha) - 2(\alpha, \rho)\), by Lemma 1.12, we have \((\alpha, \alpha) - 2(\alpha, \rho) = 0\). By Lemma 1.14 such a root must be equal to one of the \(\alpha_i\). This is a contradiction because Lemma 1.12 states that \(\alpha \neq \alpha_i\). That \(r^{-1}_1 = 0\) follows from applying \(\eta\). \(\square\)

2. The Root System

2.1 The Weyl group.

Because the reflections generating the Weyl group are defined only with respect to each \(\alpha_i\) with \(i \in I_0\), in this section much of our attention is restricted to these “real” simple roots \(\alpha_i\). As a result, many of the arguments are unchanged from the semisimple case (except that we allow infinitely many reflections \(r_i\)). We include these proofs for the sake of completeness.

Let \(R\) be the \(E\)-linear subspace of \((h^\vee)^*\) spanned by \(\Delta\), so that \(R\) has basis \(\{\alpha_i \mid i \in I\}\). For each \(i \in I_0\) define the linear transformation \(r_i : R \rightarrow R\) by:

\[ r_i \alpha_j = \alpha_j - a_{ij} \alpha_i \text{ for all } j \in I. \]

Equivalently:

\[ r_i \varphi = \varphi - \varphi(h_i) \alpha_i \text{ for all } \varphi \in R. \]

It is clear that \(r_i \alpha_i = -\alpha_i\), and \(r_i\) acts as the identity on the subspace \(\{\varphi \in R \mid \varphi(h_i) = 0\}\). The Weyl group, \(W\), is defined to be the group of linear automorphisms of \(R\) generated by the reflections \(r_i\) \((i \in I_0)\). Note that, unlike Kac-Moody algebras, generalized Kac-Moody algebras may have \(W\) generated by a set of cardinality much smaller than the number of \(\alpha_i\) \((i \in I)\).

**Definition.** A root, \(\alpha\), is defined to be real if \((\alpha, \alpha) > 0\) and imaginary if \((\alpha, \alpha) \leq 0\). Let \(\Delta_R\) denote the set of real roots and \(\Delta_I\) the set of imaginary roots. If a root \(\alpha = \sum_{i \in I} n_i \alpha_i, n_i \in \mathbb{Z}\), then define the height \(ht(\alpha) = \sum_{i \in I} n_i\).

**Proposition 2.1.** \(W\) preserves \(\Delta\), and in fact \(\dim g^\varphi = \dim g^{w \varphi}\) for \(w \in W\) and \(\varphi \in \Delta\).
It is sufficient to prove the Proposition for a reflection $r_i$. Since $i \in I_0$ (so $\alpha_i$ is real and $u_i$ is isomorphic to $\mathfrak{sl}_2$), we can apply Proposition 1.5 (8) to the $u_i$-module $\prod_{n \in \mathbb{Z}} g^{n \alpha_i}$.

\begin{proof}
Let $\varphi \in \Delta_+ \setminus \{\alpha_i\}$, then $\varphi = \sum_{j \in I} n_j \alpha_j$ where $n_j \in \mathbb{Z}$ (only finitely many $n_j$ are nonzero) and for some $j_0 \neq i, n_{j_0} > 0$. (Such a $j_0$ exists because $g^{n \alpha_i} \neq 0$ means $n = \pm 1$.) Then

$$r_i \varphi = \varphi - \varphi(h_i) \alpha_i.$$

Thus

$$r_i \varphi = \left( \sum_{j \neq i} n_j \alpha_j \right) + \left( n_i - \varphi(h_i) \right) \alpha_i,$$

and since $r_i \varphi \in \Delta$ and $n_{j_0} > 0$, it follows that $r_i \varphi \in \Delta_+$. \hfill $\square$

\end{proof}

\begin{proposition}
For all $i \in I_0$, the reflection $r_i$ permutes the elements of $\Delta_+ \setminus \{\alpha_i\}$.
\end{proposition}

\begin{proof}
Let $\varphi \in \Delta_+ \setminus \{\alpha_i\}$, then $\varphi = \sum_{j \in I} n_j \alpha_j$ where $n_j \in \mathbb{Z}$ (only finitely many $n_j$ are nonzero) and for some $j_0 \neq i, n_{j_0} > 0$. (Such a $j_0$ exists because $g^{n \alpha_i} \neq 0$ means $n = \pm 1$.) Then

$$r_i \varphi = \varphi - \varphi(h_i) \alpha_i.$$

Thus

$$r_i \varphi = \left( \sum_{j \neq i} n_j \alpha_j \right) + \left( n_i - \varphi(h_i) \right) \alpha_i,$$

and since $r_i \varphi \in \Delta$ and $n_{j_0} > 0$, it follows that $r_i \varphi \in \Delta_+$. \hfill $\square$

\end{proof}

\begin{proposition}
For all $\varphi \in \Delta_R$, $\dim g^\varphi = 1$ and in fact $\varphi \in \{W \cdot \alpha_i \mid \alpha_i \in \Delta_R \}$. Furthermore:

\begin{align*}
(2.1) & W \Delta_R = \Delta_R \\
(2.2) & W \Delta_I = \Delta_I \\
(2.3) & \Delta_R = -\Delta_R \\
(2.4) & \Delta_I = -\Delta_I \quad \text{and} \quad W(\Delta_I \cap \Delta_+) = \Delta_I \cap \Delta_+ 
\end{align*}

\end{proposition}

\begin{proof}
First we show that for all $\varphi \in \Delta_R$, $\dim g^\varphi = 1$. Let $\alpha$ be an element in $W \cdot \varphi$ such that $|\text{ht } \alpha|$ is minimal. If $\alpha = \pm \alpha_i$ for some $i$ then $\dim g^\alpha = \dim g^{\alpha_i} = 1$. Let $\alpha = \sum n_j \alpha_j$. We can assume without loss of generality that $\alpha$ is positive, i.e., each $n_j > 0$.

Since $\alpha$ is real,

$$(\alpha, \alpha) = \sum n_i (\alpha, \alpha_i) > 0.$$

Therefore,

(2.4) $$ (\alpha, \alpha_i) > 0 $$

for some $\alpha_i$. This $\alpha_i$ must be real; otherwise, $(\alpha_j, \alpha_i) \leq 0$ for all $j$, so $\sum n_j (\alpha_j, \alpha_i) = (\alpha, \alpha_i) \leq 0$ contradicting equation (2.4).

If $\alpha \neq \alpha_i$, then $r_i$ reduces the height of $\alpha$, since

$$r_i \alpha = \alpha - \alpha(h_i) \alpha_i, \quad \text{and} \quad \alpha(h_i) = \frac{(\alpha, \alpha_i)}{q_i} > 0.$$

Thus $\alpha$ cannot be of minimal height unless $\alpha = \alpha_i$ for some simple real root $\alpha_i$.

This shows $w \omega_i = \varphi$ for some $w \in W$ and by Proposition 2.1 $\dim g^\varphi = 1$.

It is easy to check that $W$ preserves the form $(\cdot, \cdot)$, so $W \Delta_R = \Delta_R$ and $W \Delta_I = \Delta_I$. The next two statements, (2.2) and (2.3), are obvious from the definitions of real and imaginary roots. To show $W(\Delta_I \cap \Delta_+) = \Delta_I \cap \Delta_+$ and
let $r_i$ be a reflection. Then as above $r_i \varphi \in \Delta_I$. Since $\alpha_i$ is real, $\varphi \in \Delta_+ \setminus \{\alpha_i\}$. By Proposition 2.2, $r_i \varphi \in \Delta_+$ so that $r_i \varphi \in \Delta_I \cap \Delta_+$. □

For all $w \in W$ define

$$\Phi_w = \Delta_+ \cap w\Delta_- = \{\psi \in \Delta_+ \mid w^{-1}\psi \in \Delta_-\}.$$ 

Note that $\Phi_w \subset \Delta_R$.

The following statements are obvious:

$$\Phi_w \subset \Delta_R \cap \Delta_+$$

$$\Phi_{w} = \emptyset$$

$$\Phi_{r_i} = \{\alpha_i\}$$

**Definition.** Given $w \in W$, let $n(w)$ be the number of elements in $\Phi_w$. Define the length of $w$, denoted $l(w)$, to be the smallest integer $k$ such that $w$ can be written as the product of $k$ of the reflections $r_i$, $i \in I_0$. An expression $w = r_{i_1}r_{i_2}\cdots r_{i_k}$, $i_j \in I$ such that $\alpha_i$ is real, is called reduced if $k = l(w)$. By convention $l(1) = 0$.

**Proposition 2.4.** Let $w \in W$ and $i \in I_0$. Then

1. $r_i(\Phi_w \setminus \{\alpha_i\}) = \Phi_{r_i w} \setminus \{\alpha_i\}$
2. If $\alpha_i \in \Phi_w$, then $\alpha_i \notin \Phi_{r_i w}$ and $n(r_i w) = n(w) - 1$. If $\alpha_i \notin \Phi_w$ then $\alpha_i \in \Phi_{r_i w}$ and $n(r_i w) = n(w) + 1$.

**Proof.** First we will show (1). Let $\varphi \in \Phi_w \setminus \{\alpha_i\}$. By Proposition 2.2, $r_i \varphi \in \Delta_+ \setminus \{\alpha_i\}$. Consider $(r_i w)^{-1}r_i \varphi = w^{-1}\varphi$. This is in $\Delta_-$ by assumption. Thus $r_i(\Phi_w \setminus \{\alpha_i\}) \subset \Phi_{r_i w} \setminus \{\alpha_i\}$, but this implies $r_i(\Phi_{r_i w} \setminus \{\alpha_i\}) \subset \Phi_w \setminus \{\alpha_i\}$ (using $r_i w$ in place of $w$). Thus equality holds.

For (2), let $\alpha_i \in \Phi_w$. Then $\alpha_i \notin \Phi_{r_i w}$ because $(r_i w)^{-1}\alpha_i = -w^{-1}\alpha_i \in \Delta_+$. By (1), $n(r_i w) = n(w) - 1$. Suppose $\alpha_i \notin \Phi_w$, then $\alpha_i \in \Phi_{r_i w}$ because $(r_i w)^{-1}\alpha_i = -w^{-1}\alpha_i \in \Delta_-$. Using (1) again, we conclude $n(r_i w) = n(w) + 1$. □

**Proposition 2.5.** For all $w \in W$

$$n(w) \leq l(w) < \infty.$$ 

**Proof.** By induction on $l(w)$. Suppose $l(w) = 1$. Then $w = r_i$, so $n(w) = 1 = l(w)$. Assume that $n(w) \leq l(w)$ for all $w \in W$ such that $l(w) = n$. Let $w \in W$ such that $l(w) = n + 1$. Then $w = r_i w'$ where $w' \in W$ has $l(w') = n$. Then by Proposition 2.4, $n(r_i w') \leq n(w') + 1$, and by the induction assumption $n(w') + 1 \leq l(w') + 1 = l(w)$. □

We will show that $l(w) = n(w)$.

**Proposition 2.6.** Let $w \in W$, $i \in I_0$, and suppose that $w\alpha_i = \alpha_j$ for some $j \in I$. Then $w r_i w^{-1} = r_j$. 

PROOF. (Note that \( r_j \) exists because \( \alpha_j \) is necessarily real.) Let \( x = r_j w r_j^{-1} \). Then \( x \) preserves \( \{ \alpha_1, \alpha_2, \ldots \} \) by the following computation: If \( \varphi \in \Delta_+ \setminus \{ \alpha_j \} \), then for some \( n \in \mathbb{Z} \)

\[
x \varphi = r_j w (w^{-1} \varphi + n \alpha_j) = r_j \varphi - n \alpha_j \in \Delta_+,
\]
as \( r_j \varphi \in \Delta_+ \setminus \{ \alpha_j \} \). Moreover, \( x \) fixes \( \alpha_j \), so \( x \) preserves \( \Delta_+ \). Thus \( x \) preserves \( \Delta_- \) as well, and so \( x \Delta_+ = \Delta_+ \). Since \( x \) is a linear automorphism it permutes \( \{ \alpha_1, \alpha_2, \ldots \} \), the set of simple roots.

For all \( k \in I \) \( x \) preserves the span of \( \alpha_j \) and \( \alpha_k \). Since \( x \alpha_k \) is a simple root, \( x \alpha_k = \alpha_k \).

PROPOSITION 2.7. Let \( w = r_{i_1} r_{i_2} \cdots r_{i_j} \), where \( i_k \in I_0 \), be a reduced expression of \( w \in W \). Then

\[
r_{i_1} r_{i_2} \cdots r_{i_{j-1}} \alpha_{i_j} \in \Delta_+.
\]

PROOF. Assume \( r_{i_1} r_{i_2} \cdots r_{i_{j-1}} \alpha_{i_j} \not\in \Delta_+ \), then choose \( m \in \{ i_1, \ldots, i_{j-1} \} \) maximal such that \( r_{i_m} r_{i_{m+1}} \cdots r_{i_{j-1}} \alpha_{i_j} \in \Delta_- \). Then \( \varphi = r_{i_{m+1}} \cdots r_{i_{j-1}} \alpha_{i_j} \in \Delta_+ \), and so \( \varphi = \alpha_{i_m} \) by Proposition 2.2. Thus setting \( y = r_{i_{m+1}} \cdots r_{i_{j-1}} \) we have \( y \alpha_{i_j} = \alpha_{i_m} \).

By Proposition 2.6,

\[
(2.5)
\]

\[
r_{i_m} y r_{i_j} = y,
\]
but this is a contradiction because the expression \((2.5)\) equals \( r_{i_m} r_{i_{m+1}} \cdots r_{i_j} \), which is reduced.

PROPOSITION 2.8. For all \( w \in W, \ l(w) = n(w) \). In fact, if \( w = r_{i_1} r_{i_2} \cdots r_{i_j} \) is a reduced expression, then \( \Phi_w \) consists of the distinct elements

\[
\alpha_{i_1}, \ r_{i_1} \alpha_{i_2}, \ r_{i_1} r_{i_2} \alpha_{i_3}, \ \ldots, \ r_{i_1} r_{i_2} \cdots r_{i_{j-1}} \alpha_{i_j}.
\]

PROOF. That all of the elements listed in the proposition are in \( \Delta_+ \) follows from applying Proposition 2.7 to \( w = r_{i_1}, w = r_{i_1} r_{i_2}, \) etc. It is easy to see that \( w^{-1} \) applied to each of the elements is in \( \Delta_- \), by using \( r_{i_n} \alpha_{i_n} = -\alpha_{i_n} \), and applying Proposition 2.7 again. If they are not all distinct, then

\[
r_{i_k} \cdots r_{i_n} \alpha_{i_{n+1}} = \alpha_{i_k} \in \Delta_+
\]
where \( r_{i_k} \cdots r_{i_n} r_{i_{n+1}} \) is some substring of \( r_{i_1} r_{i_2} \cdots r_{i_j} \). Then

\[
r_{i_{k+1}} \cdots r_{i_n} \alpha_{i_{n+1}} = -\alpha_{i_k} \in \Delta_-,
\]
which is a contradiction, because we can apply Proposition 2.7 to show

\[
r_{i_{k+1}} \cdots r_{i_n} \alpha_{i_{n+1}} \in \Delta_+.
\]
Hence by Proposition 2.5, the elements listed must exhaust \( \Phi_w \).

PROPOSITION 2.9. If \( w \in W \) preserves \( \Delta_+ \) (or equivalently, if \( \Phi_w = \emptyset \)) then \( w = 1 \).
PROOF. Apply Proposition 2.8 to the case \( n(w) = 0 \).

We extend the action of \( W \) from \( R \) to all of \((\mathfrak{h}^*)^*\). For \( i \in I_0 \), define the linear automorphism \( r'_i \) of \((\mathfrak{h}^*)^*\) by the conditions
\[
 r'_i \varphi = \varphi - \varphi(h_i)\alpha_i \quad \text{for all} \quad \varphi \in (\mathfrak{h}^*)^*
\]

Then \( r'_i|_R = r_i \), and the restriction map to \( R \) provides a homomorphism from the group \( W' \) generated by the \( r'_i \) (\( i \in I_0 \)) onto \( W \). Note that both \( r_i \) and \( r'_i \) have order 2.

**Proposition 2.10.** For all \( i \neq j \) (with \( i, j \in I_0 \)) the order \( m_{ij} \) of \( r_ir_j \) equals the order of \( r'_ir'_j \) and is related to \( a_{ij} \) as follows:

| \( a_{ij}a_{ji} \) | 0 1 2 3 4 6 \infty |
|-----------------|------------|-------------|-------------|-------------|-------------|-------------|
| \( m_{ij} \)    | 2 3 4 6 \infty |

**Proof.** The matrix of \( r_ir_j \) and also of \( r'_ir'_j \) on the 2-dimensional space \( S = \mathbb{E}\alpha_i \oplus \mathbb{E}\alpha_j \) is
\[
 T = \begin{pmatrix}
 a_{ij}a_{ji} - 1 & a_{ij} \\
 -a_{ji} & -1
\end{pmatrix}
\]
with characteristic polynomial:
\[
 (2.6) \quad \lambda^2 + (2 - a_{ij}a_{ji}) \lambda + 1.
\]

If \( a_{ij} = a_{ji} = 0 \), then \( T = -I \), and so has order 2. If \( a_{ij}a_{ji} = 1 \), then the polynomial \( (2.6) \) is \( \lambda^2 + \lambda + 1 \), hence \( T \) has order 3. If \( a_{ij}a_{ji} = 2 \), then \( (2.6) \) is \( \lambda^2 + 1 \), and has roots \( \pm \sqrt{-1} \) and hence \( T \) has order 4. If \( a_{ij}a_{ji} = 3 \), then \( (2.6) \) is \( \lambda^2 - \lambda + 1 \) with roots two primitive 6th roots of one, which means \( T \) has order 6. If \( a_{ij}a_{ji} > 4 \), then \( T \) has infinite order because the characteristic polynomial (writing \( a = a_{ij}a_{ji} \)) has roots \( \frac{1}{2} \left( a - 2 \pm \sqrt{a(a - 4)} \right) \), which are real and not \( \pm 1 \). If \( a_{ij}a_{ji} = 4 \) the characteristic polynomial is \( (\lambda - 1)^2 \), but \( T \neq 1 \), so \( T \) has infinite order.

Now we must show that the order of \( T \) on \( S \) equals the order of \( T \). If \( a_{ij}a_{ji} \neq 4 \) then the matrix
\[
 \begin{pmatrix}
 \alpha_i(h_i) & \alpha_i(h_j) \\
 \alpha_j(h_i) & \alpha_j(h_j)
\end{pmatrix}
\]
is nonsingular, so the annihilator of \( h_i \) and \( h_j \) in \( R \) (or in \((\mathfrak{h}^*)^*)\) is a linear complement \( S' \) of \( S \) in \( R \) (or in \((\mathfrak{h}^*)^*)\). But \( r_ir_j \) (or \( r'_ir'_j \)) is 1 on \( S' \); thus the order of \( T \) equals the order of \( r_ir_j \) (or \( r'_ir'_j \)). In the case \( a_{ij}a_{ji} = 4 \) then \( r_ir_j \) (or \( r'_ir'_j \)) must have infinite order because its restriction to \( S \) does.

Recall the definition of Coxeter group from [Bo]; note that it allows infinitely many generators.

**Proposition 2.11.** The restriction map
\[
 W' \to W
\]
\[
r'_i \mapsto r_i
\]
is an isomorphism. \( W \) is a Coxeter group generated by the set \( \{r_i\}_{i \in I_0} \) with defining relations \( r_i^2 = 1 \) and \( (r_ir_j)^{m_{ij}} = 1 \) (\( i \neq j \)), where \( m_{ij} \) is the order of \( r_ir_j \), given in Proposition 2.10, and \( (r_ir_j)^\infty = 1 \) is interpreted to be the vacuous relation.
PROOF. It is sufficient to prove the “exchange condition”, even in this case where the group $W$ is not finitely generated [Bo]. The “exchange condition” is:

For every $w \in W$, every reduced expression $w = r_{i_1} \cdots r_{i_l}$ ($i_k \in I_0$) and every $i \in I_0$ with $l(r_i w) < l(w)$, there exists $k \in \{1, \ldots, l\}$ such that

$$r_i r_{i_1} \cdots r_{i_{k-1}} = r_{i_1} \cdots r_{i_k}.$$

We use induction on $l(w)$. If $l(w) = 1$ then the condition is clearly true. Assume the condition for all $w$ such that $l(w) < j$. Let $w' = r_{i_1} \cdots r_{i_{j-1}}$. If $l(r_i w') < l(w')$, then we can apply the induction hypothesis to $w'$ since $l(w') = j - 1$ and we are done. Otherwise, $l(r_i w') = j$ (see Propositions 2.4 and 2.8). But

$$n(r_i w) = l(r_i w) < l(w) = n(w),$$

which implies (Proposition 2.4) $\alpha_i \in \Phi_w$ and $r_i (w')^{-1} \alpha_i \in \Delta_-$. Similarly,

$$n(r_i w') = l(r_i w') > l(w') = n(w'),$$

which implies $\alpha_i \not\in \Phi_w$, so that $(w')^{-1} \alpha_i \in \Delta_+$. This means $(w')^{-1} \alpha_i = \alpha_{i_j}$ (Proposition 2.3), so that $w' r_{i_j} (w')^{-1} = r_i$ by Proposition 2.6. Therefore,

$$r_i r_{i_1} \cdots r_{i_{j-1}} = r_{i_1} \cdots r_{i_j},$$

which is the exchange condition with $k = j$. \hfill \Box

Identify the groups $W \subset \text{Aut}(R)$ and $W' \subset \text{Aut}(\mathfrak{h}^*)$, and write $r_i$ for $r'_i$.

DEFINITION. Let $\rho$ be any fixed element of $(\mathfrak{h}^*)^*$ such that $\rho(h_i) = \frac{1}{2} a_{ii}$ for all $i \in I$. Note $\langle \rho, \alpha_i \rangle = \frac{1}{2} \langle \alpha_i, \alpha_i \rangle$

For every finite subset $\Phi$ of $R$ define $\langle \Phi \rangle = \sum_{\psi \in \Phi} \psi$, which is an element of $R$. Note that $\langle \Phi_w \rangle$ is defined for all $w \in W$ and that $\langle \Phi_r \rangle = \alpha_i$ for all $i \in I_0$.

PROPOSITION 2.12. For all $w \in W$

$$\langle \Phi_w \rangle = \rho - w \rho.$$

In particular $\rho - w \rho$ is a nonnegative integral linear combination of (finitely many of) the (real) $\alpha_i$, $i \in I_0$.

PROOF. By induction on $l(w)$. The result is clear for $l(w) = 0$ or 1. Suppose that $\langle \Phi_w \rangle = \rho - w \rho$ for all $w' \in W$ with $l(w') < l(w)$. Let $w = r_{i_1} \cdots r_{i_l}$ be a reduced expression for $w$ and let $w' = r_{i_2} \cdots r_{i_l}$, then $l(w') = l(w) - 1$. Now,

$$\rho - w \rho = \rho - r_{i_1} w' \rho = r_{i_1}(\rho - w' \rho) - r_{i_1} \rho = r_{i_1} \langle \Phi_{w'} \rangle + \langle \Phi_{r_{i_1}} \rangle$$

and by Proposition 2.4 this equals $\langle \Phi_w \rangle$. \hfill \Box

PROPOSITION 2.13. For all $w_1, w_2 \in W$

$$\langle \Phi_{w_1 w_2} \rangle = \langle \Phi_{w_1} \rangle + w_1 \langle \Phi_{w_2} \rangle.$$. 
PROOF. This follows immediately from Proposition 2.12.

PROPOSITION 2.14. The only Weyl group element that fixes $\rho$ is the identity. Equivalently, if $w_1 \rho = w_2 \rho$ where $w_1, w_2 \in W$, then $w_1 = w_2$. 

PROOF. If $w \rho = \rho$ for $w \in W$ then $\langle \Phi_w \rangle = 0$ by Proposition 2.12. Then by Proposition 2.9, $w = 1$. 

PROPOSITION 2.15. If $w_1, w_2 \in W$ and $\Phi_{w_1} = \Phi_{w_2}$ or if $\langle \Phi_{w_1} \rangle = \langle \Phi_{w_2} \rangle$ then $w_1 = w_2$. 

PROOF. Proposition 2.12 implies that if $\langle \Phi_{w_1} \rangle = \langle \Phi_{w_2} \rangle$ then $\rho - w_1 \rho = \rho - w_2 \rho$. So $w_1 \rho = w_2 \rho$. Thus by Proposition 2.14, $w_1 = w_2$. 

2.2 Some inequalities involving the root system. 

DEFINITIONS. 
1. Let $\lambda \in (b^*)^*$; then $\lambda$ is called integral if $\lambda(h_i) \in \mathbb{Z}$ when $i \in I_0$ (i.e., when $a_{ii} > 0$). The set of integral weights will be denoted by $(b^*)_I^\mathbb{Z}$. 
2. Let $\lambda \in (b^*)^*$ be such that $\lambda$ takes values in $\mathbb{F}$ on the $h_i$, for $i \in I_0$. We say $\lambda$ is in the distinguished Weyl chamber if $\lambda(h_i) \geq 0$, or equivalently if $(\lambda, \alpha_i) \geq 0$ for all such $i$. This distinguished Weyl chamber will denoted $D$. 
3. An element $\lambda \in (b^*)^*$ taking values in $\mathbb{F}$ on the $h_i$, $i \in I$, is dominant if $\lambda(h_i) \geq 0$, or equivalently if $(\lambda, \alpha_i) \geq 0$ for all $i \in I$. Denote by $P_\lambda$ the set of dominant integral elements. 

REMARK. 
1. All roots are integral (since all simple roots are integral). 
2. $W$ preserves the set of integral elements. 
3. $\rho$ is in $D$, and is integral, but $\rho$ may not be dominant. 
4. All imaginary simple roots are in $-D$. 

Note that the form $(\cdot, \cdot)$ on $(b^*)^*$ is $W$-invariant. Note that the condition that $\lambda$ takes values in $\mathbb{F}$ on the $h_i$ means that assertions such as $\lambda(h_i) \geq 0$ and $(\lambda, \alpha_i) \geq 0$ are meaningful. 

DEFINITION. Given $\mu \in P_\lambda$, for $n > 0$, let $\Omega(\mu, n)$ denote the set of all sums of $n$ distinct pairwise orthogonal imaginary simple roots, each orthogonal to $\mu$. For $n = 0$ set $\Omega(\mu, 0) = 0$. Let $\Omega(\mu) = \cup_{n \geq 0} \Omega(\mu, n)$. For $\eta \in \Omega(\mu)$, let $\{\eta\}$ denote the simple roots appearing in the sum $\eta$. 

LEMMA 2.16. Let $\psi$ be a collection of not necessarily distinct pairwise orthogonal simple imaginary roots. Let $\varphi$ be a collection of not necessarily distinct positive roots. If $\langle \psi \rangle = \langle \varphi \rangle$, then $\psi = \varphi$. 

PROOF. Let $\psi = \{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$, and let $\varphi = \{\varphi_1, \ldots, \varphi_r\}$. We show there is a bijection between $\{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$ and $\{\varphi_1, \ldots, \varphi_r\}$. Assume that $\langle \psi \rangle = \langle \varphi \rangle$. If $\nu$ is any subset of $\psi$ containing at least two elements, then $\langle \nu \rangle \notin \Delta_+$ by Lemma 1.13 (and the fact that $n \alpha_i \in \Delta_+$ implies that $n = \pm 1$). In particular, $\langle \nu \rangle \neq \varphi_j$ for all $j$. Thus $k \leq r$. 


Furthermore, because the simple roots are linearly independent, no collection of elements in \( \varphi \subset \Delta_+ \) can sum to an \( \alpha_i \) unless one of the elements equals \( \alpha_i \) and the rest are zero. Thus \( \langle \psi \rangle = \langle \varphi \rangle \) implies \( \psi = \varphi \). \( \square \)

It will be useful to distinguish between imaginary roots that can be written as \( w\alpha_i \) for some \( w \in W \) and \( i \in I \setminus I_0 \), and those that cannot. In what follows, roots of the form \( w\alpha_i \) for some \( w \in W \) and \( i \in I \) often assume the role played by the real roots in arguments involving Kac-Moody algebras.

**Proposition 2.17.** Let \( w \in W \), \( \Phi \) a finite subset of \( \Delta_+ \) and \( \eta \in \Omega(0) \). Let \( \gamma \in R \) be a finite sum of not necessarily distinct positive imaginary roots in \( \Delta_I \setminus \{ \sigma \alpha_i \mid \sigma \in W, i \in I \setminus I_0 \} \). If \( \langle \Phi_w \rangle + w\eta = \langle \Phi \rangle + \gamma \) then \( \gamma = 0 \) and \( \Phi = \Phi_w \cup w\{\eta\} \).

(Here \( w\{\eta\} \) has the obvious meaning.) In particular, \( \Phi \) consists of Weyl group transforms of simple roots.

**Proof.** Let \( \beta_1, \ldots, \beta_n \) be the distinct elements of \( \Phi \). Let \( \gamma_1, \ldots, \gamma_k \) be the distinct elements of \( \Phi_w \) and let \( \alpha_1, \ldots, \alpha_j \) be the distinct elements of \( \{\eta\} \). The proposition is clear if both \( k \) and \( j \) are zero, so assume \( k \neq 0 \) or \( j \neq 0 \).

**Case 1.** Assume that \( w^{-1}\beta_i \in \Delta_+ \) for all \( i = 1, \ldots, n \). Then

\[
w^{-1}\gamma_1 + \cdots + w^{-1}\gamma_k + \alpha_1 + \cdots + \alpha_j = w^{-1}\beta_1 + \cdots + w^{-1}\beta_n + w^{-1}\gamma.
\]

Equivalently,

\[
\alpha_1 + \cdots + \alpha_j = -w^{-1}\gamma_1 - \cdots - w^{-1}\gamma_k + w^{-1}\beta_1 + \cdots + w^{-1}\beta_n + w^{-1}\gamma.
\]

By the assumption on the \( \beta_i \), \( i = 1, \ldots, n \), and the definitions of \( \Phi_w \) and \( \gamma \) the above expression is of the form

\[
\alpha_1 + \cdots + \alpha_j = \varphi_1 + \cdots + \varphi_r,
\]

where each \( \varphi_i \in \Delta_+ \). Note that not all of the \( \varphi_i \) might not be distinct, because of the definition of \( \gamma \).

Thus by Lemma 2.16

\[
\{\eta\} = \{\varphi_1, \ldots, \varphi_r\} = \{-w^{-1}\gamma_1, \ldots, -w^{-1}\gamma_k, w^{-1}\beta_1, \ldots, w^{-1}\beta_n\} \cup w^{-1}\{\gamma\}.
\]

By the definition of \( \gamma \), \( w^{-1}\{\gamma\} \cap \{\eta\} = \emptyset \). Thus \( \gamma = 0 \). Furthermore, since the elements \( \gamma_m \) (\( m = 1, \ldots, k \)) of \( \Phi_w \) are real, \( \{\eta\} \cap \{-w^{-1}\gamma_1, \ldots, -w^{-1}\gamma_k\} = \emptyset \), so \( \{\alpha_1, \ldots, \alpha_j\} = \{w^{-1}\beta_1, \ldots, w^{-1}\beta_n\} \cap \{\gamma_1, \ldots, \gamma_k\} = \emptyset \). Thus \( \langle \Phi_w \rangle = 0 \), \( w = 1 \) and \( \{\alpha_1, \ldots, \alpha_j\} = \{\beta_1, \ldots, \beta_n\} \).

**Case 2.** Assume that some \( \beta_i \in \Phi \) satisfies \( w^{-1}\beta_i \in \Delta_- \). Then \( \beta_i \in \Phi_w \), so \( \beta_i = \gamma_s \) for some \( s, 1 \leq s \leq k \). Now consider the sum

\[
\langle \Phi_w \setminus \{\gamma_s\} \rangle + w\eta = \langle \Phi \setminus \{\beta_i\} \rangle + \gamma.
\]

If all of the remaining \( \beta_i \) satisfy \( w^{-1}\beta_i \in \Delta_+ \), then we apply the argument of Case 1 above (except that \( w \neq 1 \) here). This establishes \( \gamma = 0 \), \( \{w\alpha_1, \ldots, w\alpha_j\} = \{\beta_1, \ldots, \beta_n\} \setminus \{\beta_i\} \) and \( \{\gamma_1, \ldots, \gamma_k\} \setminus \{\gamma_s\} = \emptyset \). Otherwise, there is some \( \beta_r \) such that \( w^{-1}\beta_r \in \Delta_- \), i.e., \( \beta_r = \gamma_t \) for some \( t, 1 \leq t \leq k \) and we consider
\[ \langle \Phi_w \setminus \{ \gamma_n, \gamma_t \} \rangle + w^\eta = \langle \Phi \setminus \{ \beta_i, \beta_t \} \rangle + \gamma. \] We iterate this procedure until there are no more elements \( \beta_i \) of \( \Phi \) satisfying \( w^{-1}\beta_i \in \Delta_+ \), and \( \Phi_w \) is exhausted. \( \square \)

**Proposition 2.18.** Let \( T \) be the subset of \( (h^c)^* \) consisting of the elements of the form \( -\langle \Phi \rangle - \gamma \) where \( \Phi \) is a finite subset of \( \Delta_+ \) and \( \gamma \) is a finite sum of not necessarily distinct positive imaginary roots in \( \Delta_I \setminus \{ \sigma \alpha_i \mid \sigma \in W, i \in I \setminus I_0 \} \). Then \( \rho + T \) is \( W \)-invariant.

**Proof.** It suffices to show that \( r_i(\rho - \langle \Phi \rangle - \gamma) - \rho \in T \) for all reflections \( r_i \). If \( \alpha_i \in \Phi \) then \( \Phi = \Phi' \cup \{ \alpha_i \} \) where \( \Phi' = \Phi \setminus \{ \alpha_i \} \). We have \( \rho(h_i) = 1 \) and

\[
\begin{align*}
r_i(\rho - \langle \Phi \rangle - \gamma) - \rho &= \rho - \rho(h_i)\alpha_i - r_i\langle \Phi' \rangle + \alpha_i - r_i\gamma - \rho \\
&= -r_i(\langle \Phi' \rangle - r_i\gamma) \in T
\end{align*}
\]

by Proposition 2.2 and 2.3. If \( \alpha_i \notin \Phi \) then

\[
r_i(\rho - \langle \Phi \rangle - \gamma) - \rho = -\alpha_i - r_i(\langle \Phi \rangle) - r_i\gamma \in T
\]
as above. \( \square \)

**Proposition 2.19.** Let \( \lambda \in (h_c)^* \) be integral, and let \( U \) be a \( W \)-invariant subset of \( (h^c)^* \), all of whose elements are of the form \( \lambda - \sum_{i \in I} n_i \alpha_i \), \( n_i \in \mathbb{N} \). Then every element of \( U \) is conjugate to an element in \( D \).

**Proof.** Let \( \mu \in U \). Choose \( w \in W \) so that in the expression \( w\mu = \lambda - \sum_{i \in I} n_i \alpha_i \) the sum \( \sum_{i \in I} n_i \) is minimal. Then \( w\mu \) is in \( D \) because if \( (w\mu)(h_i) < 0 \) for some real \( \alpha_i \), then \( r_i w\mu = w\mu - m\alpha_i \), where \( m = w\mu(h_i) < 0 \). Since \( U \) is \( W \)-invariant \( r_i w\mu = \lambda - \sum_{j \in I} n_j \alpha_j - m\alpha_i \) and \( n_i + m < n_i \). So \( r_i w\mu = \lambda - \sum_{j \in I} m_j \alpha_j \) where \( \sum_{i \in I} m_i < \sum_{i \in I} n_i \), which is a contradiction. \( \square \)

**Proposition 2.20.** Let \( \mu \in (h^c)^* \) be dominant and let \( \nu = \mu - \sum_{i \in I} n_i \alpha_i \), where \( n_i \in \mathbb{N} \). If \( \nu + \rho \) is in \( D \), then

\[
(\mu + \rho, \mu + \rho) - (\nu + \rho, \nu + \rho) \geq 0.
\]

Furthermore, equality holds if and only if \( \nu = \mu - \gamma \), where \( \gamma \) is a sum of not necessarily distinct pairwise orthogonal simple imaginary roots, each of which is orthogonal to \( \mu \).

**Proof.** First we show \( (\mu + \rho, \mu + \rho) - (\nu + \rho, \nu + \rho) \geq 0 \). Consider

\[
(\mu + \rho, \mu + \rho) - (\nu + \rho, \nu + \rho) = (\mu + \rho, \mu + \rho) - (\mu + \rho - \sum n_i \alpha_i, \mu + \rho - \sum n_i \alpha_i) \]

\[
= (\mu, \sum n_i \alpha_i) + (\nu + 2\rho, \sum n_i \alpha_i).
\]

By the assumption on \( \mu \), we have \( (\mu, \sum n_i \alpha_i) \geq 0 \). So it is sufficient to show

\[
(\nu + 2\rho, \sum n_i \alpha_i) \geq 0.
\]
Let $\alpha_k$ be a root appearing in $\mu - \nu$, so $n_k > 0$ in the expression $\nu = \mu - \sum_{i \in I} n_i \alpha_i$. If $\alpha_k \in \Delta_R$ then $n_k(\nu + \rho + \rho, \alpha_k) \geq 0$, because $\nu + \rho$ and $\rho$ are in $D$.

Thus we have the desired inequality contributing to (2.9) for real $\alpha_k$.

If $\alpha_k \in \Delta_I$ then, by definition of $\rho$, $(\nu + 2\rho, \alpha_k) = (\nu + \alpha_k, \alpha_k)$ and

$$\nu + \alpha_k = \mu - \sum_{i \in I} n_i \alpha_i + \alpha_k$$

$$= \mu - \sum_{i \in I} m_i \alpha_i$$

where $n_k \geq 1$ and $m_i \in \mathbb{N}$. So

$$(\nu + \alpha_k, \alpha_k) = (\mu - \sum_{i \in I} m_i \alpha_i, \alpha_k)$$

(2.10)

$$= (\mu, \alpha_k) - \sum_{i \in I} m_i (\alpha_i, \alpha_k) \geq 0$$

by assumption on $\mu$, and since $\alpha_k \in \Delta_I$. Thus $(\nu + 2\rho, \alpha_k) = (\nu + \alpha_k, \alpha_k) \geq 0$ for $\alpha_k \in \Delta_I$ and (2.9) holds for imaginary $\alpha_i$. Summing over real and imaginary roots gives the inequality (2.9). This in conjunction with equation (2.8) gives the desired result.

Equality holds if and only if $n_i(\mu + \nu + 2\rho, \alpha_i) = 0$ for all $i \in I$, since all of the terms in (2.9) have the same sign. If $\alpha_i \in \Delta_R$ we have $n_i(\rho, \alpha_i) = \frac{n_i(\alpha_i, \alpha_i)}{2} \geq 0$, $n_i(\nu + \rho, \alpha_i) \geq 0$ and

$$n_i(\mu + \nu + 2\rho, \alpha_i) = n_i(\mu, \alpha_i) + n_i(\nu + \rho, \alpha_i) + n_i(\rho, \alpha_i)$$

$$= 0$$

which implies $\frac{n_i(\alpha_i, \alpha_i)}{2} = 0$, so that $n_i = 0$. Thus no real roots $\alpha_i$ appear in the expression

$$\nu = \mu - \sum_{i \in I} n_i \alpha_i.$$  

If $\alpha_i \in \Delta_I$ then we have from the previous calculation (2.10):

(2.11)

$$n_i[(\mu, \alpha_i) + (\mu, \alpha_i) - \sum_{j \in I} m_j (\alpha_j, \alpha_i)] = 0,$$

where $m_j = n_j$ if $i \neq j$ and $m_i = n_i - 1$. If $n_i > 0$ then (2.11) holds if and only if the conditions

1. $(\mu, \alpha_i) = 0$
2. for all $j \neq i$, $n_j > 0$ implies $(\alpha_j, \alpha_i) = 0$
3. if $n_i > 1$, then $(\alpha_i, \alpha_i) = 0$

are satisfied.

Thus $\nu = \mu - \sum_{i \in I} n_i \alpha_i$ where the $\alpha_i$’s appearing satisfy the statement in the theorem. This condition is clearly necessary and sufficient. $\square$
Proposition 2.21. Let $T \subset (\mathfrak{h}^e)^*$ be as in Proposition 2.18, let $\mu \in (\mathfrak{h}^e)^*$ be dominant integral and let $T'$ be a $W$-invariant subset of $(\mathfrak{h}^e)^*$ all of whose elements are of the form $\mu - \sum_{i \in I} n_i \alpha_i$ where $n_i \in \mathbb{N}$, excluding the elements (other than $\mu$) of the form $\mu - \sum n_i \alpha_i$ where $(\mu, \alpha_i) = 0$ if $n_i > 0$. Suppose $\lambda = \tau + \tau'$ with $\tau \in T$, $\tau' \in T'$. Then

$$(\mu + \rho, \mu + \rho) - (\lambda + \rho, \lambda + \rho) \geq 0.$$ 

Equality holds if and only if there exists $w \in W$ and an $\eta \in \Omega(\mu)$ such that $\lambda + \rho = w(\mu + \rho - \eta)$. Equivalently, equality holds if and only if $\tau = -\langle \Phi_w \rangle - w\eta$ and $\tau' = w\mu$. In the case of equality, both $\eta$ and $w$ are uniquely determined by $\lambda$, and so are $\tau$ and $\tau'$.

Proof. By Proposition 2.18, $\rho + T + T'$ is $W$-invariant. Also, every element of $\rho + T + T'$ is of the form $\mu + \rho - \sum_{i \in I} n_i \alpha_i$. Applying Proposition 2.19 there exists $w \in W$ such that

$$w^{-1}(\lambda + \rho) \in D \cap (\rho + T + T').$$

Then we may write

$$w^{-1}(\lambda + \rho) - \rho = \mu - \sum_{i \in I} n_i \alpha_i, n_i \in \mathbb{N}$$

and apply Proposition 2.20 to $\nu = w^{-1}(\lambda + \rho) - \rho$. So

$$(\mu + \rho, \mu + \rho) - (w^{-1}(\lambda + \rho), w^{-1}(\lambda + \rho)) \geq 0$$

which is equivalent to

$$(\mu + \rho, \mu + \rho) - (\lambda + \rho, \lambda + \rho) \geq 0$$

by $W$-invariance of $\langle \cdot, \cdot \rangle$. Equality holds if and only if $w^{-1}(\lambda + \rho) = \mu + \rho - \sum_{k \in I} n_k \alpha_k$ where $\alpha_k$ with $n_k > 0$ are simple imaginary, orthogonal to $\mu$, and $(\alpha_k, \alpha_j) = 0$ for $k \neq j$, $(\alpha_k, \alpha_k) = 0$ if $n_k > 1$. Thus equality holds only if there exists such an element $w$, and conversely, if there is such a $w$, then as in (2.8)

$$(\mu + \rho, \mu + \rho) - (\lambda + \rho, \lambda + \rho) = (2\mu + 2\rho - \sum_{k \in I} n_k \alpha_k, \sum_{k \in I} n_k \alpha_k)$$

$$= (2\rho - \sum_{k \in I} n_k \alpha_k, \sum_{k \in I} n_k \alpha_k)$$

$$= \sum_{k \in I} n_k (\sum_{i \neq k \in I} n_i (\alpha_i, \alpha_k) + (n_k - 1) (\alpha_k, \alpha_k))$$

$$= 0.$$
where $\Phi \subset \Delta_+$ and $\psi$ is a finite sum of not necessarily distinct positive imaginary roots in $\Delta_+ \setminus \{\sigma \alpha_i \mid \sigma \in W, i \in I \setminus I_0\}$. Applying Lemma 2.16 to the expression (2.12), we conclude $\psi = 0$, and that the collection consisting of the $\alpha_k$ with each $\alpha_k$ listed $n_k$ times is equal to the set $\Phi$. Since $\Phi \subset \Delta_+$ each $n_k = 0$ or 1. This shows $\gamma = \eta$ for some $\eta \in \Omega(\mu)$.

We have shown equality holds if and only if
\[ w^{-1}(\tau + \rho) - \rho = -\eta \quad \text{and} \quad w^{-1}\tau' = \mu \]
if and only if
\[ \tau + \rho - w\rho = -w\eta \quad \text{and} \quad \tau' = w\mu \]
by the $W$-invariance of $T'$ and $T + \rho$, and by the definitions of $T$ and $T'$.

It remains only to show the uniqueness of $w$ and $\eta$. Suppose now that
\[ \lambda + \rho = w'(\mu + \rho + \eta') = w(\mu + \rho + \eta) \]
(with $w'$ and $\eta'$ having the same properties as $w$ and $\gamma$). Let $w_0 = w^{-1}w'$. Consider
\[ w_0(\mu + \rho - \eta') = \mu + \rho - \eta. \]
Thus
\[(2.13) \quad w_0\mu - w_0\eta' + w_0\rho - \rho = \mu - \eta. \]
Since $\tau' = w\mu$ and $T'$ is $W$-invariant, $\mu \in T'$, so $w_0\mu \in T'$. We decompose $w_0\mu$ as
\[ w_0\mu = \mu - \sum_{i \in I} m_i \alpha_i - \sum_{j \in I} n_j \alpha_j - \sum_{k \in I} p_k \alpha_k \]
where $m_i, n_j, p_k \in \mathbb{N}$ and the $\alpha_i$ are real, the $\alpha_j$ are imaginary with $(\alpha_j, \mu) \neq 0$, and the $\alpha_k$ are imaginary with $(\alpha_k, \mu) = 0$.

Also,
\[ -w_0\eta' = -\eta' - \sum_{l \in I} q_l \alpha_l \]
with $\alpha_l$ real, $q_l \in \mathbb{Z}$. In fact, $q_l \in \mathbb{N}$ because for an imaginary simple $\alpha_m$, $w_0\alpha_m = \alpha_m + \sum s_n \alpha_n$ ($\alpha_n$ real) is a positive root, so $s_n \in \mathbb{N}$, and $\eta'$ is a sum of such $\alpha_m$s. By Proposition 2.12
\[ w_0\rho - \rho = -\langle \Phi_{w_0} \rangle = -\sum_{r \in I} t_r \alpha_r \]
with $\alpha_r$ real and $t_r \in \mathbb{N}$. Furthermore, $\eta = \sum s_l \alpha_s$ with $\alpha_s$ imaginary, $u_s \in \mathbb{N}$ and $(\alpha_s, \mu) = 0$. Equating coefficients in (2.13) gives:
\[ -\sum_{i \in I} m_i \alpha_i - \sum_{l \in I} q_l \alpha_l - \sum_{r \in I} t_r \alpha_r = 0. \]
and
\[ -\sum_{j \in I} n_j \alpha_j = 0, \]
so each $m_i = 0, q_l = 0, t_r = 0$, as well as each $n_j = 0$. Thus $w_0\mu = \mu - \sum p_k \alpha_k \in T'$, so by the definition of $T'$ each $p_k = 0$. Thus $w_0\mu = \mu, w_0\eta' = \eta$ and $w_0\rho - \rho = 0$. The latter equality gives $w_0\rho = \rho$, i.e., $w_0 = 1$. Equation (2.13) gives $\mu - \eta' = \mu - \eta$. We conclude that $w = w'$ and $\eta' = \eta$. □
3. Standard Modules and a Character Formula

We shall prove the character formula (due to Borcherds) for “standard modules”, which are defined below. The method used here for obtaining the character formula for a standard g\textsuperscript{e}-module and denominator identity for g is to generalize the results of [GL]. The arguments presented here are very close to those in [GL], additional arguments and lemmas are provided to deal with the presence of simple imaginary roots. The character and denominator formulas appear in [B1] and [K]; in both of these references the indicated proof is to follow the methods of [K]. Examples of standard modules for generalized Kac-Moody algebras with one simple imaginary root can be found in [GT], where they are called basic modules. Other examples of standard modules are found in [JLW].

3.1 Definitions and preliminary results.

We define “standard modules” for a generalized Kac-Moody algebra g\textsuperscript{e}. It is for this class of modules that a character formula can be proven using methods analogous to those used in the Kac-Moody case. The definition of standard module given here agrees with the definition of a standard module for the case when g\textsuperscript{e} is a Kac-Moody algebra. Integrable modules for a generalized Kac-Moody algebra, as they are defined in [K], that are also highest weight modules do not have to be standard modules by the definition given here.

Let X be an h\textsuperscript{e}-module, e.g., a g\textsuperscript{e}-module considered an h\textsuperscript{e}-module by restriction, and let \( \nu \in (h^e)^\ast \). Define the weight space 

\[ X_\nu = \{ x \in X | h \cdot x = \nu(h)x \text{ for all } h \in h^e \}. \]

An element \( \nu \in (h^e)^\ast \) is a weight if \( X_\nu \neq 0 \). X is called a weight module if X is a direct sum of its weight spaces.

Some examples of weight modules that we encounter are: \( g^e \) (under the adjoint operation), \( X \otimes Y \) if X and Y are weight modules, \( \bigwedge(n^+) \) and \( U(g^e) \).

Note that if X is a \( g^e \)-module, \( \varphi \in \Delta \) and \( \nu \in (h^e)^\ast \) then

\[ g^e \cdot X_\nu \subset X_{\nu + \varphi}. \]

A \( g^e \)-module is called a highest weight module if it is generated by a weight vector x which is annihilated by \( n^+ \). The vector x is called a highest weight vector and x is uniquely determined up to nonzero scalar. If x has weight \( \Lambda \) then \( \Lambda \) is called the highest weight of \( X \). For any highest weight module \( X \):

\[ U(n^-) \cdot x = X \]

\[ X = \bigoplus_{\lambda \leq \Lambda} X_\lambda, \quad X_\lambda = \mathbb{E}x, \quad \text{dim } X_\lambda < \infty \]

where \( \Lambda \) is the highest weight, x is the highest weight vector, and “\( \lambda \leq \Lambda \)” means \( \lambda \) is of the form \( \Lambda - \sum_{i \in I} n_i \alpha_i, n_i \in \mathbb{N} \). Lowest weight modules can be defined analogously.

Let \( S \subset I_0 \), define \( \Delta^S = \Delta \cap \bigsqcup_{i \in S} \mathbb{Z}\alpha_i \), \( \Delta^+_S = \Delta^S \cap \Delta^+ \) and \( \Delta_-^S = \Delta_- \cap \Delta^S \). Denote by \( h_S \) the span of the \( h_i, i \in S \). Let \( g_S \) be the subalgebra of \( g \) generated by the \( e_i \) and \( f_i \) with \( i \in S \). Note that \( g_S \) is isomorphic to the derived subalgebra
of the Kac-Moody algebra associated to the matrix \((a_{ij})_{i,j \in S}\). We have the root space decomposition:

\[
g_S = \bigoplus_{\varphi \in \Delta^S_+} g^\varphi \oplus h_S \oplus \bigoplus_{\varphi \in \Delta^S_-} g^\varphi.
\]

Define the following subalgebras of \(g = g(A)\):

\[
n^+ = \bigoplus_{\varphi \in \Delta^S_+} g^\varphi; \quad n^- = \bigoplus_{\varphi \in \Delta_-} g^\varphi; \quad n^+_S = \bigoplus_{\varphi \in \Delta^S_+} g^\varphi;
\]

\[
n^- = \bigoplus_{\varphi \in \Delta^-} g^\varphi; \quad u^+ = \bigoplus_{\varphi \in \Delta^S_+ \setminus \Delta^S_0} g^\varphi; \quad u^- = \bigoplus_{\varphi \in \Delta^- \setminus \Delta^S_0} g^\varphi; \quad r = g_S + h.
\]

Let \(p = r \oplus u^+\), the “parabolic subalgebra” of \(g\) defined by \(S\). Note that \(g = u^- \oplus p\). The subalgebras \(p^c\), \(r^c\) and \(g^c_S\) are defined in the obvious ways. Note that if \(S = \emptyset\) one obtains the usual Borel subalgebra \(b = h \oplus n^+\).

Let \(\Delta_+(S) = \Delta_+ \setminus \Delta^S_0\). Define \(W(S)\) to be the set \(\{w \in W \mid \Phi_w \subset \Delta_+(S)\}\).

Given \(\lambda \in (\mathfrak{h}^c)^*\) such that \(\lambda(h_i) \in \mathbb{Z}_+\) for \(i \in S\), define a universal highest weight module as follows: Let \(L(\lambda)\) be the (unique up to isomorphism) irreducible \(g^c_S\)-module with highest weight \(\lambda\). Then the generalized Verma module \(V^{L(\lambda)}\) with highest weight \(\lambda\) is defined to be the induced \(g^c\)-module

\[
V^{L(\lambda)} = U(g^c) \otimes u(\varphi(\lambda)) \ L(\lambda).
\]

It is clear from the definition that \(V^{L(\lambda)}\) is a highest weight module with highest weight \(\lambda\), and the highest weight space can be identified with the highest weight space of the \(g_S\)-module \(L(\lambda)\). By the Poincaré-Birkhoff-Witt theorem there is an isomorphism of vector spaces

\[
U(u^-) \otimes L(\lambda) \cong V^{L(\lambda)}.
\]

The \(g^c\)-module induced from the one-dimensional \(h^c\) weight module of weight \(\lambda\) is the Verma module \(V^\lambda\). Given any highest weight module \(X\), with highest weight \(\mu\) and highest weight vector \(x\), then there is a \(g^c\)-module surjection:

\[
V^\lambda \rightarrow X.
\]

Let \(X\) be a \(g^c\)-module. Recall \(P^+\) denotes the set of dominant integral elements (see §2.2).

**Definition.** A module \(X\) is a **standard module** if \(X\) is a highest weight module with highest weight \(\mu \in P^+\) and highest weight vector \(x\) satisfying:

1. for \(i \in I\) if \(\mu(h_i) = 0\) then \(f_i \cdot x = 0\)
2. if \(a_i\) (\(i \in I\)) is real then \(f_i^{n_i + 1} \cdot x = 0\), where \(n_i = \mu(h_i)\).

**Proposition 3.1.** Let \(\mu \in P^+\). Then there exist standard \(g^c\)-modules \(X^\mu_{\text{max}}\) and \(X^\mu_{\text{min}}\) of highest weight \(\mu\), such that given any standard module \(X\) with highest weight \(\mu\), there are \(g^c\)-module surjections

\[
X^\mu_{\text{max}} \rightarrow X \rightarrow X^\mu_{\text{min}}.
\]

The module \(X^\mu_{\text{min}}\) is the unique irreducible highest weight module with highest weight \(\mu\).
PROOF. These standard modules are constructed from Verma modules as they are for Kac-Moody algebras. Given \( \mu \in P_+ \) let

\[ J = I_0 \cup \{ i \in I \mid \mu(h_i) = 0 \} \subset I. \]

For \( i \in J \), let \( n_i = \mu(h_i) \). Let \( v_0 = 1 \otimes 1 \in V^\mu \). If \( a_{ii} > 0 \) (i.e., if \( i \in I_0 \)), then by the representation theory of \( \mathfrak{sl}_2 \),

\[ e_i \cdot (f_i^{n_i+1} \cdot v_0) = 0. \]

If \( i \in J \) is such that \( \mu(h_i) = 0 \), then

\[ e_i \cdot f_i^{n_i+1} \cdot v_0 = e_i \cdot f_i \cdot v_0 = \mu(h_i) \cdot v_0 = 0. \]

In fact, \( f_i^{n_i+1} \cdot v_0 \) is annihilated by \( n^+ \), because it is also true that if \( j \neq i \), then \( [e_j, f_i] = 0 \). Let

\[ Y = V^\mu / \sum_{i \in J} \mathcal{U}(\mathfrak{g}^e) f_i^{n_i+1} \cdot v_0; \]

then \( Y \) is a standard module of highest weight \( \mu \).

If \( W \) is the maximal submodule of \( Y \) not intersecting the highest weight space of \( Y \), then \( Z = Y/W \) is an irreducible standard highest weight module of highest weight \( \mu \).

Let \( X \) be a standard module with highest weight \( \mu \) and let \( x \) be a highest weight vector. Then \( f_i^{n_i+1} \cdot x = 0 \) for all \( i \in J \), and by the universal property of \( V^\mu \) and the definition of \( Z \) we have surjections:

\[ Y \rightarrow X \rightarrow Z. \]

Thus we can take \( Y = X^\mu_{\text{max}} \) and \( Z = X^\mu_{\text{min}} \).

The module \( Z \) is the unique (up to isomorphism) irreducible highest weight module of highest weight \( \mu \): If \( X \) is another such module, with highest weight vector \( x \), then for each \( i \in J \) the elements \( f_i^{n_i+1} \cdot x \) are \( n^+ \)-invariant (using an argument similar to the above), so each generates a proper \( \mathfrak{g}^e \)-submodule of \( X \), and must therefore be zero. Thus \( X \) is also standard. Hence \( X \) is isomorphic to \( Z \). □

DEFINITION. Let \( P_S = \{ \lambda \in (\mathfrak{h}^o)^* \mid \lambda(h_i) \in \mathbb{N} \text{ for all } i \in S \} \).

REMARK. Proposition 3.1 of [GL] holds in this setting without change, so there is a natural bijection between \( P_S \) and the set of (isomorphism classes) of irreducible finite-dimensional \( \mathfrak{r}^e \)-modules which are irreducible as \( \mathfrak{g}_S \)-modules.

PROPOSITION 3.2. The set of weights of a highest weight module \( X \) satisfying condition (2) above is stable under \( W \) acting on \( (\mathfrak{h}^o)^* \). For every weight \( \mu \) of \( X \) and every \( w \in W \), \( \dim X_{\mu} = \dim X_{w\mu} \). In particular, if \( \mu \) is the highest weight of \( X \), then \( \dim X_{w\mu} = 1 \) for all \( w \in W \).

PROOF. Let \( u_i \) be the Lie algebra isomorphic to \( \mathfrak{sl}_2 \) spanned by \( \{ e_i, f_i, h_i \} \) for \( a_{ii} > 0 \). The \( e_i \)-invariant highest weight vector \( x \) is contained in the finite-dimensional irreducible \( u_i \)-module \( V = \sum_{m \geq 0} \mathbb{E} f_i^m \cdot x \). Since \( X \) is generated by \( x \), any \( v \in X \) is in a finite dimensional \( u_i \)-module, so \( X \) is completely reducible under \( u_i \).
For any weight $\mu$ we can apply $\mathfrak{sl}_2$ theory to the $u_i$-stable space $\prod_{n \in \mathbb{Z}} X_{\mu-n\alpha_i}$ to conclude that $\dim X_{\mu} = \dim X_{\mu-\mu(h_i)\alpha_i} = \dim X_{r,\mu}$. The rest of the proposition is clear.

We will need the following lemma and corollary.

**Lemma 3.3.** Let $X$ be a standard module with highest weight $\mu$. Then the set of weights of $X$ does not include elements other than $\mu$ of the form $\mu - \sum_{i=1}^I n_i \alpha_i$ for $n_i \geq 0$ where $(\mu, \alpha_i) = 0$ if $n_i > 0$. That is, for every weight $\nu$, $\mu - \nu$ is not a nonzero linear combination of simple roots all orthogonal to $\mu$.

**Proof.** Suppose $v \in X$ such that $h \cdot v = (\mu - \sum_{i \in I} n_i \alpha_i)(h) \cdot v$ where the $\alpha_i$ and $n_i$ are as above. Then $v$ is a linear combination of elements of the form $f_{i_1}f_{i_2} \cdots f_{i_n} \cdot x$ for some $i_1, i_2, \cdots, i_n \in I$. We may assume that each of the $f_{i_k}$ and in particular, $f_{i_n}$, must correspond to one of the above $\alpha_i$’s where $n_i \neq 0$ (i.e. $[h, f_{i_k}] = -\alpha_i(h) f_{i_k}$), because $v$ has the given weight. Thus $f_{i_n}$ corresponds to some $\alpha_i$ such that $(\mu, \alpha_i) = 0$, so by assumption $f_{i_n} \cdot x = 0$. Thus each summand in $v$ is 0, so $v$ itself must be 0.

Since the set of weights of $X$ is stable under the action of $W$ we have the following corollary.

**Corollary 3.4.** The set of weights of $X$ does not include $W$-transforms of elements of the form given in the lemma.

\[ \square \]

### 3.2 The Casimir operator.

Define for $\nu \in (\mathfrak{h}^\ast)^+$

\[
D(\nu) = \left\{ \nu - \sum_{i \in I} n_i \alpha_i | n_i \in \mathbb{N}_+ \right\} \subset (\mathfrak{h}^\ast)^\ast.
\]

Let $\mathcal{O}$ be the category of $\mathfrak{g}^\ast$-modules $X$ such that $X$ is a weight module with finite-dimensional weight spaces, and whose set of weights lies in a finite union of sets of the form $D(\nu)$.

Recall from §1.3 that $\mathfrak{g}^\ast$ has a symmetric invariant bilinear form $(\cdot, \cdot)$, and that this form satisfies:

\[
[a, b] = (a, b) h_\varphi \quad \varphi \in \Delta, a \in \mathfrak{g}^\varphi, b \in \mathfrak{g}^{-\varphi}
\]

\[
(h, h_\varphi) = \varphi(h) \quad \forall h \in \mathfrak{h}^\varphi, \varphi \in R.
\]

The form $(\cdot, \cdot)$ induces a nonsingular pairing between $\mathfrak{g}^\varphi$ and $\mathfrak{g}^{-\varphi}$, (see [K] and [L] where the arguments work in this generality). Furthermore, $(\mathfrak{g}^\varphi, \mathfrak{g}^{-\varphi}) = 0$ unless $\varphi = -\psi$, $(\mathfrak{g}^\varphi, \mathfrak{h}^\psi) = 0$, and $(e_i, f_j) = \delta_{ij} \frac{1}{q}$ for $i, j \in I$.

Let \{s_1, \ldots, s_m\} be a basis of $\mathfrak{g}^\varphi$, $\varphi \in \Delta$. Then there is a unique dual basis \{t_1, \ldots, t_m\} of $\mathfrak{g}^{-\varphi}$ relative to $(\cdot, \cdot)$. Set $w_\varphi = \sum_{i=1}^m t_i s_i$, which is an element of $\mathcal{U}(\mathfrak{g}^\varphi)$, the universal enveloping algebra of $\mathfrak{g}^\varphi$.

The element $w_\varphi$ is independent of the basis \{s_i\}_{i=1}^m, because it can be written as $w_\varphi = f \circ (\kappa_\varphi \otimes 1)(\iota_\varphi)$, where $\iota_\varphi \in (\mathfrak{g}^\varphi)^* \otimes \mathfrak{g}^\varphi \simeq \text{End } \mathfrak{g}^\varphi$ is the unique element corresponding to $1 \in \text{End } \mathfrak{g}^\varphi$, where $\kappa_\varphi : (\mathfrak{g}^\varphi)^* \to \mathfrak{g}^{-\varphi}$ is the linear isomorphism
given by \((\cdot, \cdot)|_{\mathfrak{g}^* \times \mathfrak{g}^*}\) and \(f : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathcal{U}^{+}(\mathfrak{g}^+)^{+}\) is the map induced by multiplication.

It is clear that the element \(f \circ (\kappa_\varphi \otimes 1)(\iota_{\varphi})\) of \(\mathcal{U}^{+}(\mathfrak{g}^+)^{+}\) is the same as \(w_\varphi\), because with respect to the basis \(\{s_j\}_{j=1}^\rho\), \(\iota_{\varphi} = \sum_{i=1}^\rho s_i^* \otimes s_i\) where \(s_i^*\) is the dual element to \(s_i\).

On \(X \in \mathcal{O}\)

\[
\Gamma_1 = 2 \sum_{\varphi \in \Delta_{+}} w_\varphi
\]

acts as a well-defined operator, because once it is applied to some \(x \in X\) the sum becomes a finite one. Note that \(w_{\alpha_i} = q_i f_i e_i\) (these \(q_i\) are the diagonal entries of the matrix \(D\) that symmetrizes \(A\)). Define \(\Gamma_2\) on \(X \in \mathcal{O}\) by the condition that \(\Gamma_2\) acts on the weight space \(X_\rho\) as scalar multiplication by \((\rho, \rho + \rho)\).

**Definition.** The *Casimir operator* \(\Gamma_X\) is defined by \(\Gamma_X = \Gamma_1 + \Gamma_2\).

**Proposition 3.5.** Let \(X, Y \in \mathcal{O}\), \(f : X \rightarrow Y\) be a \(\mathfrak{g}^+\)-module map. Then \(f \circ \Gamma_X = \Gamma_Y \circ f\).

**Proof.** Since \(f\) is a module map it is easy to see that \(\Gamma_1 \circ f = f \circ \Gamma_1\). It is obvious that \(\Gamma_2\) commutes with \(f\). \(\square\)

**Proposition 3.6.** For \(X \in \mathcal{O}\) the Casimir operator \(\Gamma_X\) commutes with the action of \(\mathfrak{g}^+\) on \(X\).

**Proof.** It is easy to see that \(\Gamma_X h = h \Gamma_X\) for \(h \in \mathfrak{h}^+\). So it is sufficient to show that \(e_i \Gamma_X = \Gamma_X e_i\) and \(f_i \Gamma_X = \Gamma_X f_i\), for all \(i \in I\).

Let \(\Phi = \Delta_{+} \setminus \{\alpha_i\}\) be a union of \(\alpha_i\)-root strings, so that \(\Phi = \bigcup_{\varphi} \{\varphi + n_i \alpha_i | n_i \in \mathbb{Z}\}\) \(\cap \Delta_{+}\), where \(\varphi \in (\mathfrak{h}^+)^*\). Recall \(u_i\) is Lie subalgebra spanned by \(\{e_i, f_i, h_i\}\). Then \(M = \prod_{\varphi \in \Phi} \mathfrak{g}^\varphi\) and \(N = \prod_{\varphi \in -\Phi} \mathfrak{g}^\varphi\) are \(u_i\) (not necessarily \(s_i\)) modules that are contragredient under the \((\mathfrak{g}^\varphi\)-invariant) form \((\cdot, \cdot)\). The formal “canonical element” \(\sum_{\varphi \in \Phi} w_\varphi\) is formally \(u_i\)-invariant, and is a well-defined operator on any vector in \(M\), so acts as a \(u_i\)-invariant operator.

Let \(\nu \in (\mathfrak{h}^+)^*\) and let \(x \in X_\nu\). Take \(\Phi = \Delta_{+} \setminus \{\alpha_i\}\). Then

\[
\begin{align*}
\Gamma_1(x) &= 2w_{\alpha_i} x + 2 \sum_{\varphi \in \Phi} w_\varphi x = 2q_i f_i e_i x + 2 \sum_{\varphi \in \Phi} w_\varphi x \\
\Gamma_1(f_i x) &= 2q_i f_i e_i f_i x + 2 \sum_{\varphi \in \Phi} w_\varphi f_i x \\
\Gamma_1(e_i x) &= 2q_i f_i e_i^2 x + 2 \sum_{\varphi \in \Phi} w_\varphi e_i x.
\end{align*}
\]

Thus

\[
\begin{align*}
f_i \Gamma_1(x) - \Gamma_1(f_i x) &= 2q_i (f_i^2 e_i - f_i e_i f_i) x \\
&= 2q_i f_i (-h_i) x = -2q_i f_i \nu(h_i) x \\
&= -2q_i \nu(h_i) f_i x
\end{align*}
\]

and

\[
\begin{align*}
e_i \Gamma_1(x) - \Gamma_1(e_i x) &= 2q_i (e_i f_i e_i - f_i e_i^2) x \\
&= 2q_i h_i e_i x = 2q_i (e_i h_i + a_i e_i) x \\
&= 2q_i (\nu(h_i) + a_i) e_i x.
\end{align*}
\]
Now consider $\Gamma_2$:
\[
f_i\Gamma_2(x) - \Gamma_2(f_i x) = (\nu + \rho, \nu + \rho) f_i x - (\nu - \alpha_i + \rho, \nu - \alpha_i + \rho) f_i x
= (2(\nu + \rho, \alpha_i) - (\alpha_i, \alpha_i)) f_i x
= (2(\nu, \alpha_i) + (\alpha_i, \alpha_i)) f_i x
= 2\nu(x_{\alpha_i}) f_i x = 2q_i \nu(h_i) f_i x
\]
and
\[
e_i\Gamma_2(x) - \Gamma_2(e_i x) = (\nu + \rho, \nu + \rho) e_i x - (\nu + \alpha_i + \rho, \nu + \alpha_i + \rho) e_i x
= (2(\nu, \alpha_i) + (\alpha_i, \alpha_i)) e_i x
= (2(\nu(x_{\alpha_i}) + (\alpha_i, \alpha_i)) e_i x = -2q_i (\nu(h_i) + a_{ii}).
\]
Thus $f_i \Gamma_X(x) = \Gamma_X(f_i x)$ and $e_i \Gamma_X(x) = \Gamma_X(e_i x)$ for all $x \in X_\nu$ and hence for all $x \in X$.

\[\square\]

**Corollary 3.7.** Let $X$ be a highest weight module for $\mathfrak{g}^e$ with highest weight $\lambda$. Then $X$ lies in the category $\mathcal{O}$, and the Casimir operator of $X$ acts on $X$ as scalar multiplication by $(\lambda + \rho, \lambda + \rho)$. In particular, if $X$ contains an $\mathfrak{n}^+$-annihilated weight vector of weight $\eta \in (\mathfrak{h}^e)^*$, then $(\eta + \rho, \eta + \rho) = (\lambda + \rho, \lambda + \rho)$.

**Proof.** Let $x$ be the highest weight vector of $X$, let $y \in \mathfrak{g}^e$. Then $\Gamma_X \cdot (y \cdot x) = y \cdot \Gamma_X \cdot x = (\lambda + \rho, \lambda + \rho) y \cdot x$ by Proposition 3.6.

\[\square\]

### 3.3 Computation of the homology $H_\ast(u^-, X)$.

A general exposition of Lie algebra homology is contained in Section 1 of [GL]. We will apply this to the Lie algebra $\mathfrak{g}^e = \mathfrak{p}^e \oplus \mathfrak{u}^-$, where $\mathfrak{p}^e = \mathfrak{u}^+ \oplus \mathfrak{g}_S \oplus \mathfrak{h}^e$. Let $X$ be a standard $\mathfrak{g}^e$-module with highest weight $\mu \in P_+$. A $U(u^-)$-free resolution of $X$ is given below.

**Proposition 3.8.** Let $X$ be a standard $\mathfrak{g}^e$-module. For each $j \in \mathbb{N}$, let $D_j^X$ be the $\mathfrak{g}^e$-module $\mathcal{U}(\mathfrak{g}^e) \otimes_{\mathcal{U}(\mathfrak{p}^e)} (\bigwedge^j(\mathfrak{g}^e/\mathfrak{p}^e) \otimes X)$, where $X$ is regarded as a $\mathfrak{p}^e$-module by restriction. Then there is an exact sequence of $\mathfrak{g}^e$-modules and $\mathfrak{g}^e$-module maps
\[
\cdots \xrightarrow{d_j^X} D_2^X \xrightarrow{d_1^X} D_1^X \xrightarrow{d_0^X} D_0^X \xrightarrow{\epsilon} X \longrightarrow 0,
\]
with this complex naturally isomorphic to $V(\mathfrak{g}^e, \mathfrak{p}^e, X)$ (using the notation of [GL]). Each $D_j^X$ is isomorphic as an $U(u^-)$-module and as an $\mathfrak{t}^e$-module to $U(u^-) \otimes_{\mathcal{E}} (\bigwedge^j(u^-) \otimes X)$, with $U(u^-)$ acting by left multiplication on the first factor, and $\mathfrak{t}^e$ acting via the tensor product action on the tensor product of the three $\mathfrak{t}^e$-modules.

**Proof.** This follows directly from Proposition 1.9 [GL].

The $\mathfrak{t}^e$-module complex $C_\ast (x)$ given by
\[
\cdots \xrightarrow{1 \otimes d_2^X} \mathcal{E} \otimes_{\mathcal{U}(u^-)} D_2^X \xrightarrow{1 \otimes d_1^X} \mathcal{E} \otimes_{\mathcal{U}(u^-)} D_1^X \xrightarrow{1 \otimes d_0^X} \mathcal{E} \otimes_{\mathcal{U}(u^-)} D_0^X \longrightarrow 0
\]
is naturally isomorphic to the standard $r^e$ complex, see [GL], for computing the homology $H_*(u^-, X')$ as an $r^e$-module, and for each $j \in \mathbb{N}$, $C_j(X) \simeq \wedge^j(u^-) \otimes X$.

Since $X$ and all the $D_j^X$ lie in the category $\mathcal{O}$, we can form the Casimir operators $\Gamma_X$ and $\Gamma(j) = \Gamma_{D_j^X}$. For any vector space $Y$ let $I_Y$ denote the identity transformation on $Y$. Then $\Gamma_X = (\mu + \rho, \mu + \rho)I_X$ by Corollary 3.7, and Proposition 3.5 implies that

$$d_{j+1}^X \circ \Gamma(j + 1) = \Gamma(j) \circ d_j^X$$

for all $j \in \mathbb{N}_+$ and that

$$e_0^X \circ \Gamma(0) = (\mu + \rho, \mu + \rho)e_0^X.$$

By Proposition 3.6 the $\Gamma(j)$ ($j \in \mathbb{N}$) are $g^e$-module maps, so they are $U(u^-)$-module maps. Since the $D_j^X$ are $U(u^-)$-free, and form an exact sequence, there is a chain homotopy between the maps $(\mu + \rho, \mu + \rho)I_{D_j^X}$ and $\Gamma(j)$ (for $j \in \mathbb{N}$). More explicitly, we can construct $U(u^-)$-module homomorphisms $h_j : D_j^X \to D_{j+1}^X$ ($j \in \mathbb{N}$) such that

$$h_{j-1} \circ d_j^X + d_{j+1} \circ h_j = \Gamma(j) - (\mu + \rho, \mu + \rho)I_{D_j^X},$$

for all $j \in \mathbb{N}$ where if $j = 0$ the leftmost term is omitted.

Then the operator

$$1 \otimes \Gamma(j) - (\mu + \rho, \mu + \rho)I_{C_j(X)} \in \text{End} C_j(X)$$

induces the zero map on the $j^{th}$ homology $H_j(u^-, X')$ of $C_*(X)$, for each $j \in \mathbb{N}$.

As in [GL], we shall decompose the Casimir operator $\Gamma_Y$ for $Y \in \mathcal{O}$. Set

$$\Gamma_{Y,1} = 2 \sum_{\varphi \in \Delta_+(S)} w_\varphi \in \text{End} Y,$$

Then $\Gamma_{Y,1}$ commutes with the action of $r^e$ on $Y$. Thus $\Gamma_{Y,2} \in \text{End} Y$ defined by $\Gamma_{Y,2} = \Gamma_Y - \Gamma_{Y,1}$ commutes with the action of $r^e$ by Proposition 3.6.

Set $\Gamma_2(j) = \Gamma_{D_j^X,2}$ ($j \in \mathbb{N}$). Then as $r^e$-module endomorphisms of $C_j(X) = \mathbb{E} \otimes_{U(u^-)} D_j^X$,

$$1 \otimes \Gamma(j) = 1 \otimes \Gamma_2(j).$$

Thus

$$1 \otimes \Gamma_2(j) - (\mu + \rho, \mu + \rho)I_{C_j(X)}$$

induces the zero map on the $j^{th}$ homology of $C_*(X)$.

For $\lambda \in P_S$ denote by $C_j(X)(\lambda)$ the sum of all the $r^e$-submodules of $C_j(X)$ isomorphic to $L(\lambda)$. Set

$$B_j(X) = \prod_{\lambda \in P_S, (\lambda, \rho, \lambda + \rho) = (\mu + \rho, \mu + \rho)} C_j(X)(\lambda),$$

and also let

$$B'_j(X) = \prod_{\lambda \in P_S, (\lambda, \rho, \lambda + \rho) \neq (\mu + \rho, \mu + \rho)} C_j(X)(\lambda),$$

for all $j \in \mathbb{N}$. Then $C_j(X) = B_j(X) \oplus B'_j(X)$, because $C_j(X)$ is a completely reducible $g_S$-module, and $C_*(X)$ is the direct sum of the $r^e$-module subcomplexes $B_*(X)$ and $B'_*(X)$. When restricted to $C_j(X)(\lambda)$, the operator $1 \otimes \Gamma_2(j)$ acts as
the scalar $(\lambda + \rho, \lambda + \rho)$. Thus $1 \otimes \Gamma_2(j) - (\mu + \rho, \mu + \rho)I_{C_j(X)}$ is a nonsingular
operator on $B_j'(X)$, this shows:

**Proposition 3.9.** The homology of $B_j'(X)$ is zero. In particular, the homology $H_* (u^-, X^i)$ of $C_j(X)$ is naturally a $g^e$-module isomorphic to the homology of its subcomplex $B_*(X)$.

To finish the computation of $H_* (u^-, X^i)$ we must understand the weight spaces of $C_j(X)_{(\lambda, \rho)}$ and the weight spaces of $\Lambda^j (u^-)$.

**Proposition 3.10.** For each $j \in \mathbb{N}$, the weights of $\Lambda^j (u^-)$ lie in the subset $T$ of $(\mathfrak{h}^e)^*$ defined in Proposition 2.18. Let $w \in W(S)$, and let $\eta \in \Omega(0)$. Suppose that $l(w) \leq j$. If $\eta \in \Omega(j, j - l(w))$, then $-\langle \Phi_w \rangle - w\eta$ is a weight of $\Lambda^j (u^-)$ and the corresponding weight space is one-dimensional. If $w \in W$ and either $w \notin W(S)$ or $l(w) + \text{ht}(\eta) \neq j$ then $-\langle \Phi_w \rangle - w\eta$ is not a weight of $\Lambda^j (u^-)$.

**Proof.** We have the decomposition $n^- = \bigoplus_{\varphi \in \Delta_+} g^{-\varphi}$. Choose a basis $\{b_i\}_{i \in J}$ of $n^-$ such that for each $i \in J$, $b_i \in g^{-\varphi_i}$. Let $J' \subset J$ be such that $\{b_i\}_{i \in J'}$ is a basis of $u^- \subset n^-$. Since we can assume the indexing set $J$ is linearly ordered, a basis for $\Lambda^j (n^-)$ is given by $\{b_{i_1} \wedge \cdots \wedge b_{i_j} \mid i_1 < \cdots < i_j \}$ where $i_m \in J$, and a basis for $\Lambda^j (u^-)$ is $\{b_{i_1} \wedge \cdots \wedge b_{i_j} \mid i_m \in J' \}$. For each sequence $i_1 < \cdots < i_j$, $b_{i_1} \wedge \cdots \wedge b_{i_j}$ is a weight vector with weight $-\sum_{m=1}^j \varphi_{i_m}$, here $b_{i_k} \in g^{-\varphi_{i_k}}$. Since $\dim g^{-\varphi} = 1$ for any root $\varphi$ in $\{\alpha \mid w \in W, i \in I\}$ the $\varphi_{i_k}$ in $\{\alpha \mid w \in W, i \in I\}$ must be distinct so the weights are in $T$.

Let $w \in W$ with $l(w) \leq j$, then $\Phi_w = \{\varphi_{i_1}, \ldots, \varphi_{i_l(w)}\}$ for $i_1 < \cdots < i_j$ and $\varphi_{i_k} \in \Delta_+ \cap \Delta_+$ by Proposition 2.7. Let $\eta \in \Omega(j, j - l(w))$ (so $\eta = \alpha_{i_{l(w)+1}} + \cdots + \alpha_{i_l}$, where the $\alpha_{i_k}$ are $j - l(w)$ distinct simple imaginary roots, which are mutually orthogonal). Then $-\langle \Phi_w \rangle - w\eta$ is a weight of $b_{i_1} \wedge \cdots \wedge b_{i_j}$, where if $1 \leq k \leq l(w)$ then $b_{i_k}$ is in $g^{-\varphi_{i_k}}$ and if $l(w) < k \leq j$ then $b_{i_k}$ is in $g^{-\varphi_{i_k}}$. Proposition 2.17 and the one-dimensionality of the root spaces $g^{-\alpha_k}$ for $w \in W$ and $i \in I$ imply that every other basis vector of $\Lambda (n^-)$ has a weight different from $-\langle \Phi_w \rangle - w\eta$. Thus the weight space $\Lambda (n^-)_{-\langle \Phi_w \rangle - w\eta}$ has dimension one and is in $\Lambda^j (n^-)$.

If $w \in W(S)$ then $\Phi_w \subset \Delta_+ (S)$ and so $b_{i_1}, \ldots, b_{i_l(w)} \in u^-$. Furthermore, $b_{i_{l(w)+1}}, \ldots, b_{i_j} \in u^-$ by definition, so $\Lambda (n^-)_{-\langle \Phi_w \rangle - w\eta} \subset \Lambda (u^-)$.

If $w \notin W(S)$ then $\varphi_{i_m} \notin \Delta_+ (S)$ so $b_{i_m} \notin u^-$ for some $m = 1, \ldots, l(w)$. So, regardless of the choice of $\eta$, $b_{i_1} \wedge \cdots \wedge b_{i_j} \notin \Lambda^j (u^-)$ and $-\langle \Phi_w \rangle - w\eta$ is not a weight of $\Lambda^j (u^-)$. If $\text{ht} \eta \neq j - l(w)$ then we’ve seen above $-\langle \Phi_w \rangle - w\eta$ is not a weight of $\Lambda^j (u^-)$.

**Proposition 3.11.** Let $X$ be a standard $g^e$-module with highest weight $\mu \in P_+$. For all $j \in \mathbb{N}$, define $W^j$ to be the set of weights $\lambda$ of $\Lambda^j (u^-) \otimes X$ such that $(\lambda + \rho, \lambda + \rho) = (\mu + \rho, \mu + \rho)$. Set $W = \bigcup W^j$. Then there is a bijection between $W(S) \times \Omega(\mu)$ and $W$, and for each $j \in \mathbb{N}$, $W^j$ corresponds bijectively to $\{\eta \mid \eta \in W(S) \times \Omega(\mu) \mid l(w) + \text{ht}(\eta) = j\}$. Thus the sets $W^j$ are disjoint. The correspondence is given as follows: If $w \in W(S)$ and $\eta \in \Omega(\mu)$ such that
\( l(w) + ht(\eta) = j \) then \( \lambda = -(\Phi_w) - w\eta + w\mu = w(\mu + \rho - \eta) - \rho \) is the element of \( \mathcal{W}^j \) associated to the pair \((w, \eta)\).

The weight space \( \Lambda^j(\mathfrak{u}^-) \otimes X \lambda \) is one-dimensional, and is the tensor product \( \Lambda^j(\mathfrak{u}^-) - (\Phi_w) - w\eta \otimes X_{w\mu} \) of one-dimensional weight spaces.

**Proof.** Let \( w \in W(S) \) and let \( \eta \in \Omega(\mu, j - l(w)) \). Set

\[
\lambda = -(\Phi_w) - w\eta + w\mu = w(\mu + \rho - \eta) - \rho.
\]

By Proposition 3.10 \( \Lambda^j(\mathfrak{u}^-) - (\Phi_w) - w\eta \) is one-dimensional, and by Proposition 3.2 so is \( X_{w\mu} \). We have \( \Lambda^j(\mathfrak{u}^-) - (\Phi_w) - w\eta \otimes X_{w\mu} \subset (\Lambda^j(\mathfrak{u}^-) \otimes X)_\lambda \), and by construction \((\lambda + \rho, \lambda + \rho) = (\mu + \rho, \mu + \rho) \). Thus \( \lambda \in \mathcal{W}^j \) and we have a map from \( W(S) \times \Omega(\mu) \) to \( \mathcal{W}^j \) such that elements satisfying \( l(w) + ht(\eta) = j \) map to \( \mathcal{W}^j \).

Let \( T \) and \( T' \) be as in Proposition 2.21. By Proposition 3.10 the set of weights of \( \Lambda(\mathfrak{u}^-) \) lie in the subset \( T \); by Proposition 3.2 and Lemma 3.3 the set of weights of \( X \) is of type \( T' \). Let \( \lambda \in \mathcal{W}^j \). Then the weight space \( (\Lambda(\mathfrak{u}^-) \otimes X)_\lambda \) is a finite direct sum of tensor products \( \Lambda(\mathfrak{u}^-) \otimes X_\tau \) where \( \tau \in T \) is a weight of \( \Lambda(\mathfrak{u}^-) \), \( \tau' \in T' \) is a weight of \( X \), and \( \lambda = \tau + \tau' \). By Proposition 2.21 there is a unique \( w \in W \) and \( \eta \in \Omega(\mu) \) such that

\[
\lambda = w(\mu + \rho - \eta) - \rho
\]

and \( \tau = -(\Phi_w) - w\eta \) and \( \tau' = w\mu \). Since \( \tau \) is a weight of \( \Lambda(\mathfrak{u}^-) \) we see by Proposition 3.10 that in fact \( w \in W(S) \). Thus we have a bijection between \( W(S) \times \Omega(\mu) \) and \( \mathcal{W}^j \).

If we consider \( \Lambda^j(\mathfrak{u}^-) \) and \( \lambda \in \mathcal{W}^j \) in the above argument, we conclude that \( \tau = -(\Phi_w) - w\eta \) is a weight of \( \Lambda^j(\mathfrak{u}^-) \), so that \( l(w) + ht(\eta) = j \). Thus the above bijection restricts to bijections \( \{(w, \eta) \in W(S) \times \Omega(\mu) \mid l(w) + ht(\eta) = j \} \leftrightarrow \mathcal{W}^j \), for \( j \in \mathbb{N} \).

Since the decomposition \( \lambda = \tau + \tau' \) is unique

\[
\Lambda^j(\mathfrak{u}^-) - (\Phi_w) - w\eta \otimes X_{w\mu} = (\Lambda^j(\mathfrak{u}^-) \otimes X)_\lambda,
\]

where the two factors on the left are one-dimensional. \( \square \)

The \( \mathfrak{g}_S \)-module \( \Lambda^j(\mathfrak{u}^-) \otimes X \) is completely reducible, see [K]. Note that this is true even though \( S \) is not necessarily of “finite type”, as in the case of [GL], or even finite. Furthermore, by the same argument appearing in [GL], the subspace \( (\Lambda^j(\mathfrak{u}^-) \otimes X)_\lambda \) is annihilated by \( e_i \) for all \( i \in S \). In fact, the submodule generated by the one-dimensional space \( (\Lambda^j(\mathfrak{u}^-) \otimes X)_\lambda \) is isomorphic to the irreducible module \( L(\lambda) \), see also [K].

Thus we have shown the following generalization of Theorem 8.5 of [GL]:

**Theorem 3.12.** Let \( X \) be a standard \( \mathfrak{g}^e \)-module with highest weight \( \mu \in P_+ \). Let \( S \subset \mathfrak{s}_0 \). The correspondence

\[
(w, \eta) \mapsto w(\mu + \rho - \eta) - \rho
\]
is a bijection from the set \( \{ (w, \eta) \in W(S) \times \Omega(\mu) \mid l(w) + ht(\eta) = j \} \) onto the set of all weights \( \lambda \) of \( \bigwedge^j (u^-) \otimes X \) such that \( (\lambda + \rho, \lambda + \rho) = (\mu + \rho, \mu + \rho) \). Each such weight \( w(\mu + \rho - \eta) - \rho \) occurs with multiplicity one in \( \bigwedge^j (u^-) \otimes X \) with one-dimensional weight space \( \bigwedge^j (u^-) \otimes (\Phi_\omega - w \eta) \otimes X_{\omega \mu} \). Furthermore, \( w(\mu + \rho - \eta) - \rho \) is in \( P_S \), and the weight space of \( w(\mu + \rho - \eta) - \rho \) generates a copy of the irreducible \( \mathfrak{g} \)-module \( L(w(\mu + \rho - \eta) - \rho) \). In particular, in any direct sum decomposition of the \( \mathfrak{g} \)-module

\[
\bigwedge^j (u^-) \otimes X = \bigoplus_{\lambda_i} L(\lambda_i)
\]

where \( \lambda_i \in P_S \), the \( \lambda_i \) for which \( (\lambda_i + \rho, \lambda_i + \rho) = (\mu + \rho, \mu + \rho) \) must coincide with the \( w(\mu + \rho - \eta) - \rho \) as \( (w, \eta) \) ranges through \( \{ (w, \eta) \in W(S) \times \Omega(\mu) \mid l(w) + ht(\eta) = j \} \). Each such \( \lambda_i \) occurring exactly once in the decomposition \( \bigoplus_{\lambda_i} L(\lambda_i) \). In fact, the above correspondence \( (w, \eta) \mapsto w(\mu + \rho - \eta) - \rho \) is a bijection, so all of the irreducible \( \mathfrak{g} \)-modules \( L(w(\mu + \rho - \eta) - \rho) \) are inequivalent as \( w \) ranges through \( W(S) \) and \( \eta \) ranges through \( \Omega(\mu) \).

Recall the complex \( C_\ast(X) \) defined after Proposition 3.8 is the standard \( \mathfrak{g} \)-module complex for computing \( H_\ast(u^-, X^t) \). Also \( C_\ast(X) = B_\ast(x) \oplus B'_\ast(X) \) where the homology of \( B'_\ast(X) \) is zero by Proposition 3.9. The complex \( B_\ast(x) \) has all zero maps because Theorem 3.12 and Proposition 3.11 imply that each \( \lambda \) can appear as a weight of \( C_j(X) \) for only one \( j \), so that the sequence

\[
\cdots \to C_{j-1}(X)_{(\lambda)} \to C_j(X)_{(\lambda)} \to C_{j+1}(X)_{(\lambda)} \to \cdots
\]

is equal to

\[
\cdots \to 0 \to C_j(X)_{(\lambda)} \to 0 \to \cdots.
\]

Therefore \( H_j(u^-, X^t) \) can be naturally identified with the \( \mathfrak{g} \)-submodule \( B_j(X) \) of \( C_j(X) \), and

\[
B_j(X) = \bigoplus_{(\lambda + \rho, \lambda + \rho) = (\mu + \rho, \mu + \rho)} C_j(X)_{(\lambda)} = \bigoplus_{w \in W(S), l(w) \leq j \atop \eta \in \Omega(\mu, j - l(w))} L(w(\mu + \rho - \eta) - \rho).
\]

This gives:

**Theorem 3.13.** Let \( X, \mu, j \) be as in Theorem 3.12. The \( j \)-th homology space \( H_j(u^-, X^t) \) is isomorphic as an \( \mathfrak{g} \)-module to the direct sum

\[
\bigoplus_{w \in W(S), l(w) \leq j \atop \eta \in \Omega(\mu, j - l(w))} L(w(\mu + \rho - \eta) - \rho)
\]

of inequivalent irreducible \( \mathfrak{g} \)-modules.

**3.4 The character formula for standard modules.**

Let \( (\mathfrak{h}_1^c)^\ast \) denote the set of integral linear functionals. Let \( \mathcal{A} \) be the abelian group of all possibly infinite formal integral linear combinations of elements of \( (\mathfrak{h}_1^c)^\ast \). Denote by \( e^\lambda \) the element of \( \mathcal{A} \) corresponding to \( \lambda \in (\mathfrak{h}_1^c)^\ast \). Then \( \mathbb{Z}[ (\mathfrak{h}_1^c)^\ast ] \), the integral group algebra of \( (\mathfrak{h}_1^c)^\ast \), is contained in \( \mathcal{A} \), we have \( e^\lambda e^\mu = e^{\lambda + \mu} \), and elements of \( \mathcal{A} \) can be multiplied provided that the resulting expression is well defined.
Let $\mathcal{E}$ denote the category of all $\mathfrak{h}^e$-modules $X$ such that $X$ has a direct sum decomposition $\bigoplus_{\lambda \in (\mathfrak{h}^e)_+^*} X_\lambda$ such that each $X_\lambda$ is finite dimensional.

For an object $X$ in the category $\mathcal{E}$, define its formal character $X(X)$ as an element of $\mathcal{A}$ as follows:

$$X(X) = \sum_{\lambda \in (\mathfrak{h}^e)_+^*} (\dim X_\lambda) e^\lambda$$

**Lemma 3.14.** Let

$$0 \to X_1 \to X_2 \to X_3 \to 0$$

be an exact sequence in $\mathcal{E}$. Then

$$X(X_2) = X(X_1) + X(X_3).$$

**Proof.** Obvious because $\dim(X_2)_\lambda = \dim(X_1)_\lambda + \dim(X_3)_\lambda$. □

If $C_*$ is a chain complex in $\mathcal{E}$, then the homology groups $H_j$ of $C_*$ are in $\mathcal{E}$. The chain complex $C_*$ is **admissible** if $\bigoplus_j C_j$ is in $\mathcal{E}$.

**Lemma 3.15.** [Euler-Poincaré] Let $C_*$ be an admissible chain complex in $\mathcal{E}$. Then $\prod_j H_j$ is in $\mathcal{E}$ and

$$\sum_{j \in \mathbb{N}} (-1)^j X(C_j) = \sum_{j \in \mathbb{N}} (-1)^j X(H_j).$$

□

Now we apply the Euler-Poincaré principle to the chain complex $C_*$, so that $C_j = C_j(X) = \Lambda^j(u^-) \otimes X$ and $H_j = H_j(u^-, X^t)$. For the trivial one-dimensional $\mathfrak{g}^e$-module $X = \mathbb{E}$ and the special case $S = \emptyset$ (so $r^e = \mathfrak{h}^e$), Theorem 3.13 yields:

**Theorem 3.16.** We have the denominator identity

$$\prod_{\varphi \in \Delta_+} (1 - e^{-\varphi})^{\dim \mathfrak{g}^e} = \sum_{w \in \mathcal{W}} (\det w) \sum_{\eta \in \Omega(0)} (-1)^{\text{ht}(\eta)} e^{w(\rho^{-}) - \rho}$$

$$= \sum_{w \in \mathcal{W}} (\det w) \sum_{\eta \in \Omega(0)} (-1)^{\text{ht}(\eta)} e^{-\langle \Phi w \rangle - w\eta}. \quad (3.1)$$

**Proof.** There are $\dim(\Lambda^j(n^-))_\lambda = \#$ of ways of writing $\lambda$ as a sum of $j$ elements in $\Delta_+$, so that in the alternating sum, $\sum_{j \in \mathbb{N}} (-1)^j \sum_{\lambda \in (\mathfrak{h}^e)_+^*} \dim(\Lambda^j(n^-))_\lambda e^\lambda$, the coefficient of $e^\lambda = (\#$ of ways of writing $\lambda$ as the sum of an even number of elements of $\Delta_+$) - (\#$ of ways of writing $\lambda$ as an odd number of elements of $\Delta_+$). This is exactly the coefficient of $e^\lambda$ in $\prod_{\varphi \in \Delta_+} (1 - e^{-\varphi})^{\dim \mathfrak{g}^e}$. 


On the right hand side of the Euler-Poincaré formula:

\[
\sum_{j \in \mathbb{N}} (-1)^j \mathcal{X}(H_j) = \sum_{j} (-1)^j \sum_{w \in \mathbb{W}, t(w) \leq j} \sum_{\eta \in \Omega(0,j-t(w))} e^{w(\mu-\eta) - \rho}
\]

\[
= \sum_{w \in \mathbb{W}} \sum_{\eta \in \bigcup_{n \geq 0} \Omega(0, n)} (-1)^{n+t(w)} e^{w(\mu-\eta) - \rho}
\]

\[
= \sum_{\eta \in \Omega(0)} (-1)^{ht(\eta)} e^{w(\rho - \eta) - \rho}.
\]

\[
\square
\]

For \( X \) an arbitrary standard \( \mathfrak{g} \)-module with highest weight \( \mu \in \mathbb{P}_+ \) we obtain a character formula:

**Theorem 3.17.** We have the following character formula:

\[
\mathcal{X}(X) = \frac{\sum_{w \in \mathbb{W}} (\det w) \sum_{\eta \in \Omega(\mu)} (-1)^{ht(\eta)} e^{w(\mu+\rho-\eta)}}{\sum_{w \in \mathbb{W}} (\det w) \sum_{\eta \in \Omega(0)} (-1)^{ht(\eta)} e^{w(\rho-\eta)}}.
\]

**Proof.** For any standard \( \mathfrak{g} \)-module \( X \) Theorem 3.13 gives:

\[
\sum_{j \in \mathbb{N}} (-1)^j \mathcal{X}(H_j(X)) = \sum_{w \in \mathbb{W}} (\det w) \sum_{\eta \in \Omega(\mu)} (-1)^{ht(\eta)} e^{w(\mu+\rho-\eta)}.
\]

We also have:

\[
\sum_{j \in \mathbb{N}} (-1)^j \mathcal{X}(C_j(X)) = \sum_{j \in \mathbb{N}} (-1)^j \mathcal{X}(\bigwedge^j (\mathfrak{n}^-) \otimes X) = \mathcal{X}(X) \sum_{j \in \mathbb{N}} (-1)^j \mathcal{X}(C_j(X)) \]

which can be seen by writing \((\bigwedge^j (\mathfrak{n}^-) \otimes X)\lambda\) as the direct sum of spaces of the form \((\bigwedge^j (\mathfrak{n}^-))\tau \otimes X\tau'\), where \(\tau + \tau' = \lambda\). Applying the formula in Theorem 3.16, and using the Euler-Poincaré lemma, we obtain the character formula. \(\square\)

Since the character formula given in Theorem 3.17 depends only on the highest weight of the standard module \( X \), we conclude that all standard modules of the same highest weight are isomorphic. We have established:

**Corollary 3.18.** Every standard \( \mathfrak{g} \)-module is irreducible.

\(\square\)

As an application of the character formula for standard modules, one can give a short proof of Theorem 3.19, below. Theorem 3.19 is a generalization of the main theorem of [J], and the direct proof given there works in this case also, with appropriate changes (see also [JW]).

Let \( I = I_1 \cup I_2 \) (disjoint union), satisfying

1. If \( i, j \in I_2 \) and \( i \neq j \) then \( (\alpha_i, \alpha_j) \neq 0 \)
2. \( I_0 \subset I_1 \)

Let \( \mathfrak{g}_1 \) be the subalgebra of \( \mathfrak{g} \) generated by the \( e_i \) and \( f_i \) with \( i \in I_1 \). This algebra is isomorphic to the generalized Kac-Moody algebra associated to the matrix \((a_{ij})_{i,j \in I_1}\), by the results of §1.4. There is a triangular decomposition \( \mathfrak{g}_1 = \mathfrak{n}_+^1 \oplus \mathfrak{h}_1 \oplus \mathfrak{n}_-^1 \).
Theorem 3.19 (Jurisch-Wilson). Let \( \mathfrak{g}(A) = \mathfrak{g} \) be a generalized Kac-Moody algebra, associated to a symmetrizable matrix \( A \). Then \( \mathfrak{g} = u^+ \oplus (\mathfrak{g}_1 + \mathfrak{h}) \oplus u^- \), where \( u^- \) is the free Lie algebra on the direct sum of the standard highest weight \( \mathfrak{g}_1 \)-modules \( U(\mathfrak{n}_1^-) \cdot f_j \) for \( j \in I_2 \) and \( u^+ \) is the free Lie algebra on the direct sum of the standard lowest weight \( \mathfrak{g}_1 \)-modules \( U(\mathfrak{n}_1^+) \cdot e_j \) for \( j \in I_2 \).

Proof. The subalgebra \( \mathfrak{g}_1 \) of \( \mathfrak{g} \) and hence its universal enveloping algebra \( U(\mathfrak{g}_1) \) act via the adjoint action on \( \mathfrak{g} \). We identify the root \( -\alpha_j \) of \( \mathfrak{g} \) with the weight of the highest weight vector \( f_j \) of the standard \( \mathfrak{g}_1 \)-module \( U(\mathfrak{g}_1) \cdot f_j \subset \mathfrak{g} \), which we denote by \( L_1(-\alpha_j) \).

By our assumptions on \( I_2 \), the set \( \Omega(0) \) of all sums of mutually orthogonal distinct imaginary simple roots of \( \mathfrak{g} \) can be written as the disjoint union of the subset consisting of elements of \( \Omega(0) \) having no terms \( \alpha_j \) with \( j \in I_2 \), and the subset consisting of elements of \( \Omega(0) \) where there is exactly one term \( \alpha_j \) with \( j \in I_2 \). Denote by \( \Omega_1(\mu, n), n \geq 0 \), the set of sums of \( n \) distinct, pairwise orthogonal, imaginary simple roots of \( \mathfrak{g}_1 \), each orthogonal to \( \mu \in P_+ \). Then as in the character formula, let \( \Omega_1(\mu) = \cup_n \Omega_1(\mu, n) \). Then \( \Omega(0) = \bigcup_{j \in I_2} (\alpha_j + \Omega_1(-\alpha_j)) \cup \Omega_1(0) \). Here \( \alpha_j + \Omega_1(-\alpha_j) = \{ \alpha_j + \eta \mid \eta \in \Omega_1(-\alpha_j) \} \).

Since \( I_0 \subset I_1 \) we can decompose the right side of equation (3.1), the denominator identity, (call this \( D \)) of \( \mathfrak{g} \) as

\[
D = \sum_{w \in W} (\det w) \sum_{\eta \in \Omega(0)} (-1)^{ht(\eta)} e^{w(\rho - \eta) - \rho} \tag{3.3}
- \sum_{w \in W} (\det w) \sum_{j \in I_2, \eta \in \Omega_1(-\alpha_j)} (-1)^{ht(\eta)} e^{w(\rho - \eta - \alpha_j) - \rho}.
\]

The second term of equation (3.3) is precisely \( e^{-\rho} \) times the sum of the “numerators”

\[
\sum_{w \in W} (\det w) \sum_{\eta \in \Omega(-\alpha_j)} (-1)^{ht(\eta)} e^{w(-\alpha_j + \rho - \eta)}
\]

appearing in the character formula (3.2) of the standard highest weight \( \mathfrak{g}_1 \)-modules \( L_1(-\alpha_i) \) as \( j \) ranges through \( I_2 \). Let

\[
D_{\mathfrak{g}_1} = \sum_{w \in W} (\det w) \sum_{\eta \in \Omega(0)} (-1)^{ht(\eta)} e^{w(\rho - \eta) - \rho},
\]

the “denominator” (times \( e^\rho \)) of the character formula for a standard \( \mathfrak{g}_1 \)-module. Then equation (3.3) becomes

\[
D = D_{\mathfrak{g}_1} \left[ 1 - \sum_{j \in I_2} \mathcal{A}(L_1(-\alpha_j)) \right].
\]

Let \( F(V) \) denote the free Lie algebra over the vector space \( V = \bigcup_{j \in I_2} L_1(-\alpha_j) \). Let \( \Delta_+^1 \) be the set of positive roots of \( \mathfrak{g}_1 \), which can be considered as a subset of \( \Delta_+ \). By a generalization of Witt’s formula for computing the dimension of a homogeneous subspace of a graded free Lie algebra (see [J] or [Ka]) we have:

\[
1 - \sum_{j \in I_2} \mathcal{A}(L_1(-\alpha_j)) = \prod_{\varphi \in \Delta_+ \setminus \Delta_+^1} (1 - e^{-\varphi})^{\dim F(V)/\varphi}.
\]
Thus \( \dim F(V)^{-\varphi} = \dim g^{-\varphi} \). The inclusion \( V \to \bigsqcup_{\varphi \in \Delta_+ \setminus \Delta_+^1} g^{-\varphi} \) extends to a grading preserving Lie algebra homomorphism from \( F(V) \) to \( \bigsqcup_{\varphi \in \Delta_+ \setminus \Delta_+^1} g^{-\varphi} \). The free Lie algebra \( F(V) \) is isomorphic to the ideal of \( g_0^- \) (see Proposition 1.1) generated by the \( f_j \) with \( j \in I_2 \). Furthermore this ideal maps onto \( \bigsqcup_{\varphi \in \Delta_+ \setminus \Delta_+^1} g^{-\varphi} \) by construction of \( g \). Thus the grading preserving homomorphism is a surjection, and by considering dimensions we conclude that there is an isomorphism between \( F(V)^{-\varphi} \) and \( g^{-\varphi} \).

Thus there is an isomorphism

\[
\frak n^- = \frak n_1^- \ltimes \frak u^-,
\]

where \( \frak u^- = F(V) \) (the semidirect product is the obvious one given by the action of \( \frak n_1^- \) on \( \frak u^- \)). □

By iterating this procedure one can decompose \( \frak n^- \) further, eventually obtaining a decomposition involving a series of free subalgebras and a Kac-Moody subalgebra (see also [JW]).

References

Department of Mathematics, The University of Chicago, Chicago IL 60637
E-mail address: jurisich@math.uchicago.edu